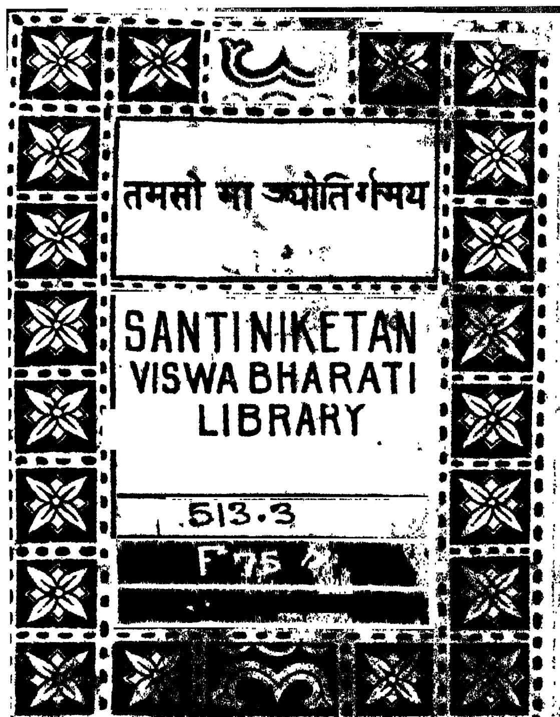




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PREFACE

This book contains the Chapters on Solid Geometry from the Plane and Solid Geometry by the same authors. The general nature of the motives that led to the organization of the work are described in the preface of the complete edition, and it does not seem necessary to repeat all of them here.

In order to make it possible to refer to theorems proved in Plane Geometry, a complete syllabus of them, together with other necessary quotations, is printed on pages xxix–xlvi of this book. All references made in the text, and any other questions in Plane Geometry concerning which there may be doubt, can there be looked up by the student. An excellent opportunity for a review of Plane Geometry is afforded by this syllabus.

The book is distinguished by its acceptance of the principle of emphasis of important theorems laid down by the Committee of Fifteen of the National Education Association in their Report.* Thus, theorems of the greatest value and importance are printed in bold-faced type, and those whose importance is considerable are printed in large italics.

The Report just mentioned has been of great assistance, and its principles have been accepted in general, not in a slavish sense but in the broad manner recommended by the Committee itself. A perusal of the Report will give more fully and accurately than could be done in this brief preface, the considerations which led to the adoption of these principles, in particular, the principle of emphasis upon important theorems, both by the Committee and by the authors of this book.

* Printed as a separate pamphlet with the *Proceedings* for 1912. Reprinted also in *School Science*, 1911, and in *The Mathematics Teacher*, December, 1912.

The great excellence of the figures, particularly the very unusual and effective 'phantom' halftone engravings, deserves mention. These figures should go far toward relieving the unreality which often attaches to the constructions of Solid Geometry in the minds of students.

W. B. FORD,
CHAS. AMMERMAN,
E. R. HEDRICK, Editor.

CONTENTS

The complete text for the theorems listed in the Syllabus of Theorems of Plane Geometry in this book is published separately under the title

PLANE GEOMETRY

The contents of that book and of this one are published together under the title

PLANE AND SOLID GEOMETRY



SOLID GEOMETRY

CHAPTER VI

LINES AND PLANES IN SPACE

PART I. GENERAL PRINCIPLES

239. Definitions. **Solid Geometry**, or **Geometry of Three Dimensions**, treats of figures whose parts are not confined to a plane.

A **plane** is a surface such that if any two points in it are taken, the straight line passing through them lies wholly in the surface.

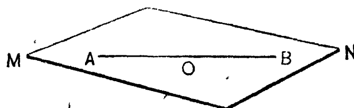


Fig. 163

Thus, in Fig. 163, if A and B are two points of a plane MN, the entire straight line AB lies in the plane MN. Any point O on AB lies in MN.

A plane is said to be **determined** by certain points and lines if that plane and no other plane contains those points and lines.

240. Corollary 1. It is evident from the definition of a plane that *if a line has two of its points in a plane, it lies wholly in that plane.*

241. Assumptions, or **Postulates.**

1. *A plane is unlimited in extent.*

2. *Through any straight line an unlimited number of planes may be passed.* See Fig. 164.

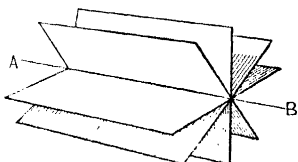


FIG. 164

3. *If a plane is revolved about any straight line in it as an axis, it may be made to pass through any point in space.*

4. *One and only one plane can be made to pass through three points not in the same straight line.*

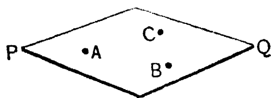


FIG. 165 (*a*)

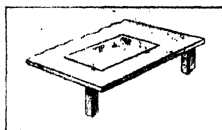

FIG. 165 (*b*)

Fig. 165 (*a*) represents a plane *PQ* through three points *A*, *B*, *C*. Fig. 165 (*b*) represents a plane piece of glass resting on the points of three tacks.

5. *Two planes cannot intersect each other in only a single point.*

242. Corollary 1. *A plane is determined by two intersecting lines.*

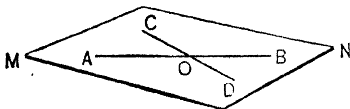
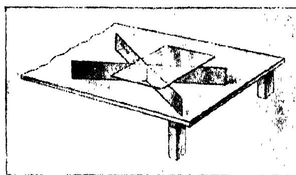


FIG. 166 (*a*)


FIG. 166 (*b*)

[HINT. Consider the point where the lines intersect, and two other points, one on each line; then apply 4, § 241.]

243. Corollary 2. *A line and a point without the line determine a plane.*

[HINT. Use 4, § 241.]



FIG. 167 (*a*)

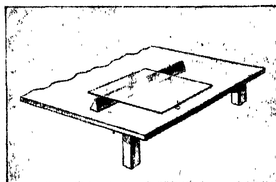


FIG. 167 (*b*)

244. Corollary 3. *Two parallel lines determine a plane.*

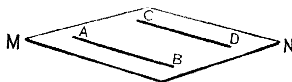


FIG. 168 (*a*)

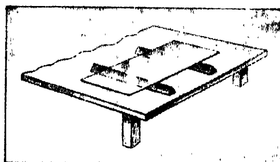


FIG. 168 (*b*)

[HINT. By the definition of parallel lines (§ 48), two such lines *must* lie in a plane. Show that this is the only one.]

EXERCISES

1. How many planes pass through a given straight line in space? How many pass through *two* given points?

2. In a carpenter's plane the knife-edge lies along a straight line. As soon as any rough surface has been sufficiently planed off, the whole length of the knife-edge keeps on the surface as the plane is moved along. Connect this fact with § 240.

3. Why are cameras, surveyors' transits, etc., mounted on three legs instead of four?

4. Prove that a straight line can intersect a plane in but one point unless it lies wholly in the plane. See § 240.

245. Theorem I. *The intersection of two planes is a straight line.*

Given the two intersecting planes MN and RS.

To prove that their intersection is a straight line.

Proof. Let A and B be any two points common to both planes. 5, § 241

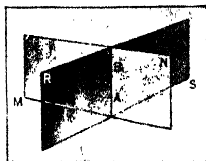


FIG. 169

Draw the straight line AB.

Then every point in AB lies in MN and also in RS. § 240

Therefore, AB is common to the two planes.

Moreover, no point not on AB can be common to both planes, for the two planes would then coincide. 4, § 241

Therefore, the intersection of the planes MN and RS is a straight line.

EXERCISES

1. What is the *locus* of all points common to two intersecting planes?

2. If a sheet of paper is folded, why is the crease straight?

3. In how many points (in general) will three planes intersect? What can be said of the intersection of four or more planes in space?

4. Can two pencils be held in such a position that a plane cannot be passed through them? State the *general* fact about a plane that is illustrated by your answer.

5. Can a plane be passed (in general) through four or more given points in space? Can a plane be passed (in general) through three lines all of which pass through a common point in space?

6. Can there be two straight lines that are not parallel and that do not meet? Find a pair of such lines in Fig. 169.

PART II. PERPENDICULARS AND PARALLELS

246. Line Perpendicular to a Plane. The point where a line intersects a plane is called the **foot** of the line on that plane.

A straight line is **perpendicular** to a plane when it is perpendicular to every straight line in the plane drawn through its

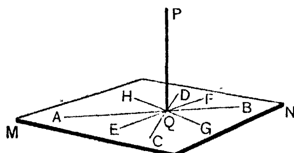
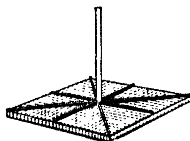


FIG. 170 (*a*)


FIG. 170 (*b*)

foot. The plane is then also said to be perpendicular to the line. Thus, in Fig. 170 (*a*), if *PQ* is perpendicular to the plane *MN*, it is then perpendicular to all the lines *QA*, *QB*, *QC*, etc.; and *PQ* is called the **distance** from *P* to *MN*; see Ex. 1 below.

247. Parallel Planes and Lines. A straight line is parallel to a plane if they never meet, however far produced. Two planes are parallel if they never meet, however far produced.

It is to be remembered (§ 48) that two lines are parallel only when they *lie in the same plane* and do not meet.

248. Corollary 1. *A plane that contains one of two parallel lines is parallel to the other line.*

EXERCISES

1. Show, by § 77, that the perpendicular from a point *P* to a plane *MN* (Fig. 170 *a*) is shorter than any other line that can be drawn from *P* to *MN*.

2. Show, by § 71, that if two oblique lines from a point *P* to a plane *MN* cut off equal distances from the foot of the perpendicular from *P* to *MN*, they are equal. See Ex. 1, p. 63.

249. Theorem II. ***If a line is perpendicular to each of two lines at their point of intersection, it is perpendicular to their plane.***

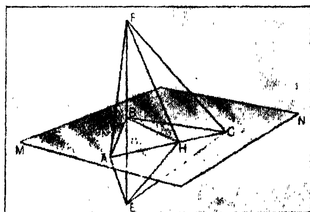


FIG. 171

Given FB perpendicular at B to each of two straight lines AB and BC of the plane MN.

To prove FB perpendicular to the plane MN.

Proof. Draw AC, and through B draw any line, as BH, meeting AC at H.

Prolong FB to E so that $BE = FB$. Join F and E to A, H, and C.

Then AB and BC are perpendicular bisectors of FE,		Const.
whence	$FA = AE$, $FC = CE$.	§ 100
Therefore	$\triangle AFC \cong \triangle AEC$,	§ 45
whence	$\angle HAF = \angle HAE$.	Why ?
Also	$\triangle HAF \cong \triangle HAE$,	Why ?
and	$HF = HE$.	Why ?
Hence	$HB \perp FE$ or FB.	Why ?
But HB was any line in MN drawn through B.		
Therefore	$FB \perp MN$.	§ 246

250. Corollary 1. *At a point in a plane only one perpendicular line can be erected.*

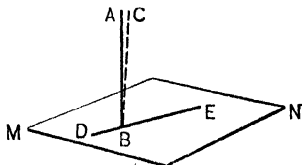


FIG. 172

[HINT. Suppose a second perpendicular line BC could be erected (Fig. 172). Pass a plane through AB and BC. This plane will intersect MN in a straight line, as DE. Then AB and BC are both perpendicular to DE at the same point B. But, since AB, BC, and DE all lie in the same plane this is impossible, by 7, § 31.]

251. Corollary 2. *From a point without a plane, only one line can be drawn perpendicular to the plane.*

[HINT. If two perpendiculars, as PB and PA, could be drawn from P to the plane MN, then $\triangle PBA$ would contain two right angles so that the sum of the angles of $\triangle PBA$ would be more than two right angles. But this is impossible. Why?]

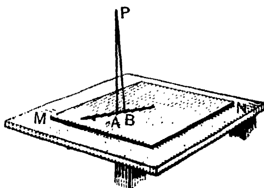


FIG. 173

252. Corollary 3. *Through a given point in a straight line, only one plane can be drawn perpendicular to the line.*

[HINT. Draw two different perpendiculars in space to the given line at the given point, and apply §§ 242, 249. If two such planes exist, their intersections with a plane through the given line violate 7, § 31.]

253. Corollary 4. *Through a given point without a straight line, only one plane can be drawn perpendicular to the line.*

[HINT. Prove by reduction to an absurdity. Show that the intersections of *two* such perpendicular planes with the *plane* determined by the given line and given point would violate § 58.]

254. Corollary 5. *All perpendicular lines that can be drawn to a straight line at a given point in it lie in a plane perpendicular to the line at the given point.*

[HINT. Show that otherwise two perpendicular lines could be drawn to the given line in the same plane at the given point, thus violating 7, § 31.]

EXERCISES

1. Show how to determine a perpendicular to a plane by means of two carpenter's squares.

2. Tell how to test whether or not a flagpole is erect.

3. A spoke of a wheel is perpendicular to the axis on which it turns. Show by § 254 that it describes a plane in its rotation.

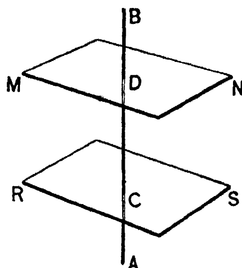


Fig. 174

255. Theorem III. *Two planes perpendicular to the same line are parallel.*

[Hint. Show that if the two planes met, say in a point P, § 253 would be violated.]

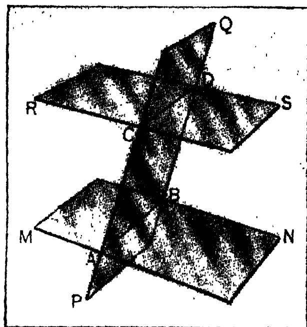


Fig. 175

256. Theorem IV. *If a plane intersects two parallel planes, the lines of intersection are parallel.*

Given the plane PQ intersecting the parallel planes MN and RS in AB and CD, respectively.

To prove $AB \parallel CD$.

[Hint. Prove, by reduction to an absurdity, that AB and CD cannot meet.]

257. Theorem V. *Two lines parallel to a third line (in space) are parallel to each other.* Compare § 50.

[Hint. Let BB' and CC' be two lines parallel to a third line AA' (Fig. 176). The plane determined by CC' and the point B on BB' is parallel to AA' (§ 248). Therefore (§ 48) the line of intersection of this plane with the plane of the parallels AA' and BB' is parallel to AA'. Hence show, by § 49, that this line of intersection coincides with BB', so that BB' and CC' lie in a plane. Finally, show that BB' and CC' cannot meet; for, if they did meet, say at a point D, the plane determined by D and AA' would contain (§ 244) both BB' and CC'.]

258. Theorem VI. *If two angles, not in the same plane, have their sides respectively parallel and extending in the same direction, they are equal and their planes are parallel.*

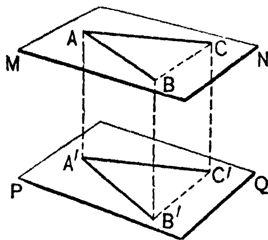


FIG. 176

Given the angles BAC and $B'A'C'$, lying in the planes MN and PQ, respectively, with $AB \parallel A'B'$, and $AC \parallel A'C'$.

To prove that $\angle A = \angle A'$, and that $MN \parallel PQ$.

Proof. Take $AB = A'B'$, and $AC = A'C'$.

Draw AA', BB', CC', CB, and $C'B'$.

Since AB is equal and parallel to $A'B'$,
it follows that $ABB'A'$ is a parallelogram; Why?
hence AA' is equal and parallel to BB'. Why?

Similarly, AA' is equal and parallel to CC'.

Hence BB' is equal and parallel to CC'. § 257

Then $BB'C'C$ is a parallelogram, and $CB = C'B'$. Why?

Therefore $\triangle ABC \cong \triangle A'B'C'$. Why?

Hence $\angle A = \angle A'$. Why?

Now $PQ \parallel AB$. Likewise $PQ \parallel AC$. § 248

Therefore, $PQ \parallel MN$ for, if not, the line of intersection of PQ and MN would meet either AB or AC (or both) extended; hence PQ would not be parallel to each of them.

NOTE. The similar theorem for angles that lie in the same plane was proved in § 67. As in § 67, the two angles are *supplementary* to each other if one pair of corresponding sides extend in *opposite* directions from the vertices.

259. Theorem VII. *A plane perpendicular to one of two parallel lines is perpendicular to the other also.*

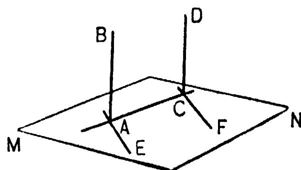


FIG. 177

Given the two parallel lines AB and CD, and a plane MN perpendicular to CD at C.

To prove that MN is perpendicular to AB.

Proof. The parallel lines AB and CD determine a plane (§ 244) which intersects MN in some line AC. § 245

Now AC is perpendicular to CD; § 246
whence AC is perpendicular to AB. § 60

Draw any line AE in the plane MN through A.

Draw CF in MN parallel to AE through C.

Then CF is perpendicular to CD. § 246

Hence AE is perpendicular to AB. § 258

Therefore AB is perpendicular to MN. § 246

260. Corollary 1. *Two lines perpendicular to the same plane are parallel.*

[HINT. Let AB and CD (Fig. 177) be perpendicular to the plane MN. Imagine a parallel CD' to AB through C. Then CD' is perpendicular to MN, by § 259. Hence CD' coincides with CD, by § 250.]

EXERCISES

1. The legs of a table lie along parallel lines in space. What preceding theorem or corollary is illustrated here? Mention other similar illustrations.

2. How many lines can be drawn through a given point parallel to a given plane? If there is more than one such, what is the locus of them all?

3. Given a plane and two points without it. When will the line through the two points be parallel to the plane?

261. Theorem VIII. *If two straight lines are intersected by three parallel planes, the corresponding segments of these lines are proportional.*

Given the straight lines AB and CD cut by the parallel planes L, M, and N.

To prove that

$$AE/EB = CF/FD.$$

Proof. Draw BC meeting the plane M in G. Draw EF, EG, FG, BD, and AC.

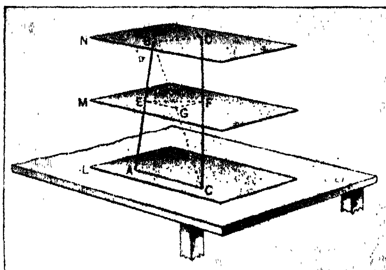


FIG. 178

Then $GF \parallel BD$, and $EG \parallel AC$. § 256

Now $AE/EB = CG/GB$, and $CF/FD = CG/GB$. § 145

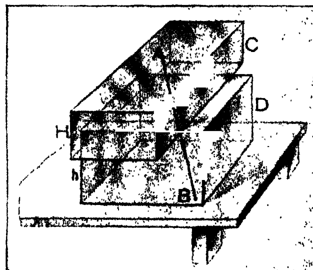
Therefore $AE/EB = CF/FD$. Why?

EXERCISES

1. Show that if parallel planes intercept equal segments on one line, they will intercept equal segments on any other line.

2. In Fig. 178, $AE = 5$, $EB = 4$, and $CF = 6$. What is the value of FD?

3. Two ordinary blocks C and D having the respective heights H and h are placed upon each other as shown in the figure. Show that any line AB drawn from the upper surface of C to the lower surface of D will be divided in the ratio $H : h$ by the point P where AB intersects the common surface of the two blocks.



262. Perpendicular Planes. Two planes *MN* and *PQ* are said to be **perpendicular** to each other when any line *CD* drawn in the one perpendicular to their intersection is perpendicular to the other plane.

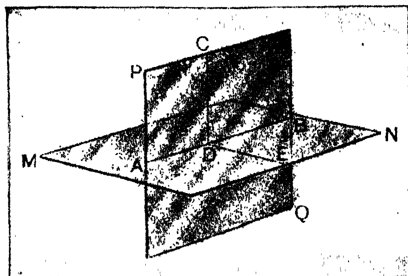


FIG. 179.

263. Theorem IX. *If a straight line is perpendicular to a plane, every plane containing this line is perpendicular to the given plane.*

Given the line *CD* perpendicular to plane *MN*; and given any plane *PQ* containing the line *CD*.

To prove that plane *PQ* is perpendicular to plane *MN*.

Proof. Let *AB* be the line of intersection of the two planes *MN* and *PQ*. Imagine any line *C'D'* in the plane *PQ* perpendicular to the line *AB*.

Then	*CD* is parallel to *C'D'*.	§ 52
But	*CD* is perpendicular to *MN*;	Given
hence	*C'D'* is perpendicular to *MN*.	§ 259

Since *C'D'* is any line of the plane *PQ* perpendicular to *AB*, it follows that *PQ* is perpendicular to *MN*. § 262

264. Corollary 1. *The line perpendicular to a given plane at a given point lies in any plane through that point perpendicular to the given plane.*

265. Theorem X. *If each of two intersecting planes is perpendicular to a third plane, their line of intersection is perpendicular to the third plane.*

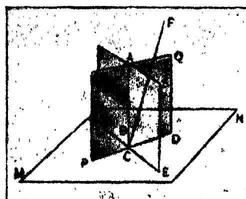


FIG. 180

Given the planes PQ and RS perpendicular to plane MN and intersecting each other in AB.

To prove that AB is perpendicular to MN.

Proof. Suppose that AB is not perpendicular to MN, but that some other line as CF through C, the point common to the three planes, is the perpendicular to MN.

Then CF lies in RS and in PQ. § 264

Hence CF coincides with AB. § 245

Therefore AB is perpendicular to MN at C.

EXERCISES

1. The blades of a side paddle wheel of a steamboat are all perpendicular to the side of the boat. Connect this fact with one of the preceding theorems. Do the same with the fact that the upright edge of any building is vertical.

2. How many planes can be drawn perpendicular to a given plane and passing through a given line in space?

[HINT. Select a point in the given line, draw the perpendicular line through that point to the given plane, and consider all the planes that can be passed through this perpendicular.]

PART III. DIHEDRAL ANGLES

266. Dihedral Angles. The figure formed by two intersecting portions of planes bounded by their line of intersection is called a **dihedral** angle. The planes forming the dihedral angle are its **faces** and the line of intersection is its **edge.**

A dihedral angle may be designated by the two letters on its edge, or by the two letters on its edge together with an additional letter on each face.

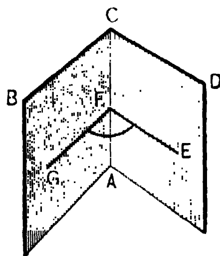


Fig. 181

Thus, in the figure, the planes *AD* and *AB* are the faces and *AC* is the edge of the dihedral angle *B–CA–D.*

The **plane angle of a dihedral angle** is an angle formed by lines in the two faces perpendicular to the edge at the same point. Thus, *GFE* is the plane angle of the dihedral angle *B–CA–D.*

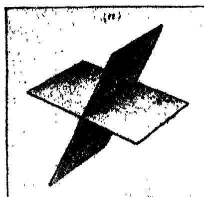
The magnitude of a dihedral angle does not depend upon the extent of its faces. If a plane be made to revolve from the position of one face about the edge as an axis to the position of the other face, it turns through the dihedral angle, and the greater the amount of turning, the greater the angle.

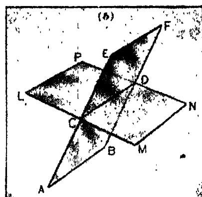
267. Measure of Dihedral Angles. The plane angle of a dihedral angle is taken as its **measure,** so that two dihedral angles are always in the same ratio as the magnitudes of their plane angles. In particular, two dihedral angles are **equal** when their plane angles are equal.

Dihedral angles are right, acute, obtuse, etc., according as their plane angles are right, acute, obtuse, etc. Similar definitions exist for complementary dihedral angles, supplementary dihedral angles, vertical dihedral angles, etc. The faces of a right dihedral angle are perpendicular to each other.

EXERCISES

1. Read the adjacent dihedral angles in the following figure. Read the vertical, the complementary, the supplementary dihedral angles.



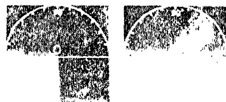


2. If two planes intersect each other, show that the opposite or vertical dihedral angles thus formed are equal.

[Hint. Use § 267.]

3. Show that the dihedral angle through which a door is opened is measured by the plane angle through which the bottom edge of the door moves.

4. Make an instrument for measuring dihedral angles by cutting and folding a piece of heavy paper or cardboard in the manner shown in the figure.



5. What is the number of degrees in one of the dihedral angles of a bay window, it being understood that the bay window consists of three equal upright plane sections, and that their bases form three sides of a regular octagon?

268. Theorem XI. *Every point in a plane that bisects a dihedral angle is equidistant from the faces of the angle.*

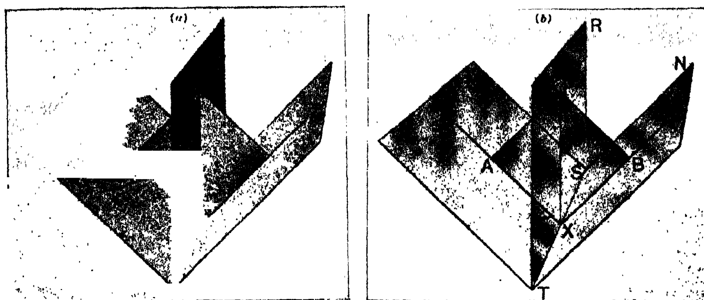


FIG. 182

Given the plane TR bisecting the dihedral angle formed by the planes TM and TN, so that the dihedral angles $M\text{–}TS\text{–}R$ and $N\text{–}TS\text{–}R$ are equal; and **given** PA and PB perpendicular to TM and TN, respectively, from any point P in TR.

To prove that $PA = PB$.

Proof. Pass a plane through PA and PB and let X be its point of intersection with ST; let AX, BX, and PX be the intersections of plane PAB with planes TM, TN, and TR.

Plane $PAB \perp$ planes TM and TN. § 263

Then plane $PAB \perp ST$, § 265

whence $ST \perp AX$, BX, and PX. Why?

The angles AXP and BXP are the plane angles of the dihedral angles $M\text{–}ST\text{–}R$ and $N\text{–}ST\text{–}R$. Why?

Since the dihedral angles are given equal, their plane angles are equal, that is, $\angle AXP = \angle BXP$;

whence rt. $\triangle AXP \cong$ rt. $\triangle BXP$, Why?

and therefore $PA = PB$. Why?

269. Corollary 1. *Any point not in the bisecting plane of a dihedral angle is unequally distant from the two faces.*

270. Corollary 2. *The plane bisecting a dihedral angle is the* **locus** *of all points equally distant from the faces of the angle.*

See NOTE, § 99.

EXERCISES

1. To what theorem in Plane Geometry does § 268 correspond?

2. From any point within a dihedral angle perpendiculars are drawn to the faces. Show that the angle formed by these perpendiculars is supplementary to the plane angle of the dihedral angle.

3. Prove that the two adjacent dihedral angles formed by one plane meeting another are supplementary.

[HINT. At some point on the edge of the dihedral, erect a plane perpendicular to its edge, and consider the plane angles formed.]

4. What is the locus of *all* points equidistant from two intersecting planes, each of indefinite extent?

5. What is the locus of all points in space equidistant from two given points?

6. What is the locus of all points in space equidistant from the circumference of a circle?

7. What is the locus in space of all points equidistant from two intersecting lines?

8. What is the locus of all points equally distant from two parallel lines?

9. Prove that of the dihedral angles formed by a plane intersecting parallel planes, the alternate and corresponding angles are equal, and the interior angles on the same side of the transversal plane are supplementary.

10. Prove that all plane angles of a dihedral angle are equal.

PART IV. POLYHEDRAL ANGLES

271. Polyhedral Angles. The figure formed by three or more straight line segments that end in a common point, together with the V-shaped portions of planes determined by pairs of adjacent lines, is called a **polyhedral angle.**

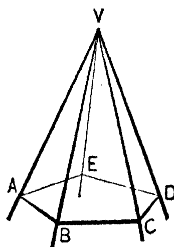


FIG. 183. POLYHEDRAL ANGLE

The point at which the lines all meet is called the **vertex** of the angle.

The lines in which the planes meet are its **edges**; and the V-shaped portions of the planes between these edges are its **faces.**

The plane angles in the faces at the vertex are called the **face angles** of the polyhedral angle.

A polyhedral angle is *read* by naming the vertex and a point in each edge. Thus, in Fig. 183, the polyhedral angle is read $V\text{–}ABCDE$.

Two polyhedral angles are **congruent** if they can be placed so that their vertices coincide and their corresponding edges coincide.

A **trihedral angle** is a polyhedral angle that has three faces.

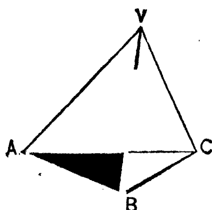


FIG. 184. TRIHEDRAL ANGLE

Thus, in Fig. 184, the three planes VAB, VBC, VAC, which meet at V form the trihedral angle $V\text{–}ABC$.

Two trihedral angles are **congruent** if the three face angles of the one are equal, respectively, to the three face angles of the other, and are arranged in the same order. This can be shown by methods similar to those of § 45. See also §§ 361, 374.

If the intersections of a plane with all the faces of a polyhedral angle is a convex polygon, the polyhedral angle is a *convex polyhedral angle.*

272. Theorem XII. *The sum of two face angles of a trihedral angle is greater than the third.*

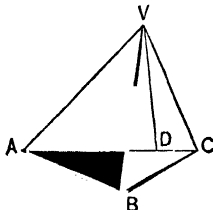


FIG. 185

Given the trihedral angle $V\text{-}ABC$.

To prove that $\angle AVB + \angle BVC > \angle AVC$.

Proof. If $\angle AVC$ is equal to or less than either of the other angles, we know the proposition is true without further proof.

If $\angle AVC$ is greater than either of the other angles, lay off any lengths VA and VC on the sides of $\angle AVC$, and draw AC. Then draw VD in the plane AVC, making $\angle AVD = \angle AVB$.

Lay off $VB = VD$, and draw AB and CB.

Then	$\triangle AVB \cong \triangle AVD$.	Why?
Therefore	$AB = AD$.	Why?
Now	$AB + BC > AD + DC$.	Why?
Whence, subtracting,	$BC > DC$.	Ax. 6
Therefore	$\angle BVC > \angle DVC$.	§ 79
By construction	$\angle AVB = \angle AVD$.	
Adding,	$\angle AVB + \angle BVC > \angle AVC$.	

EXERCISES

1. If in the trihedral angle $V\text{-}ABC$, $\angle AVB = 60°$, and $\angle BVC = 80°$, make a statement as to the number of degrees in $\angle AVC$.

2. Show that any face angle of a trihedral angle is greater than the difference of the other two.

273. Theorem XIII. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*

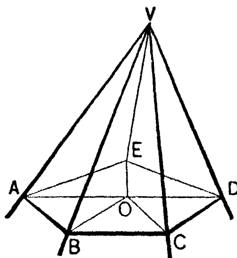


Fig. 186

Given the polyhedral angle $V\text{-}ABCDE$ with the edges cut by any plane in the points A, B, C, D, E.

To prove that the sum of the face angles of the polyhedral angle is less than four right angles.

Proof. Connect any point O in the polygon $ABCDE$ with the vertices A, B, C, D, E.

The number of triangles with the common vertex O is the same as the number having the vertex V.

Now $\angle VBA + \angle VBC > \angle ABO + \angle OBC$, § 272

and $\angle VAB + \angle VAE > \angle BAO + \angle OAE$, etc.

Therefore the sum of the base angles of the triangles having V for a common vertex is greater than the sum of the base angles of the triangles having O for vertex.

But the sum of all the angles of all the triangles whose vertex is V is equal to the sum of all the angles of all the triangles whose vertex is O. Why?

Therefore the sum of the angles about the vertex V is less than the sum of the angles about O, that is, less than four right angles.

MISCELLANEOUS EXERCISES ON CHAPTER VI

1. Lean one book in a slanting position against another book that lies flat on a table, and hold a stretched string parallel to the cover of the slanting book. Can the string have more than one position? Can the string be horizontal? Vertical?

2. Show that if a half-open book is placed on a table, resting on its bottom edges, the back edge of the book is perpendicular to the plane of the table, and the lines of printing are parallel to that plane.

3. Show that the dihedral angle between the pages of an open book is measured by the plane angle between opposite lines of type on the two pages.

4. What is the shape of the end of an ordinary plank after it has been sawed off in a slanting direction, assuming that the opposite faces of the original board are parallel planes?

5. Prove that the segments of two parallel lines included between parallel planes are equal.

[HINT. Pass a plane through the parallel lines and then prove that the given segments form the opposite sides of a parallelogram.

6. Prove that a plane perpendicular to the edge of a dihedral angle is perpendicular to both its faces.

[HINT. Use § 263.]

7. What is the locus of all the points equidistant from the three faces of a trihedral angle?

8. Show that the locus of any given point on a line segment of fixed length, whose ends touch two parallel planes, is a third plane parallel to the given planes.

9. Prove that if three lines are perpendicular to each other at a common point in space, each line is perpendicular to the plane of the other two, and that the planes of the lines (taken in pairs) are perpendicular to each other. Note how this is illustrated on a cube, or in a corner of a room, or in a corner of an ordinary box.

10. The trihedral angle formed when three planes meet each other, so that each is perpendicular to the other two is called **a trirectangular** trihedral angle.

Prove that the edges of a trirectangular trihedral angle are mutually perpendicular by pairs. See §§ 246, 265.

11. Prove that the space about a point is divided into eight congruent trirectangular trihedral angles by three planes mutually perpendicular by pairs at the point.

12. Prove that if a line is parallel to the intersection of two planes, it is parallel to each of the planes.

[HINT. Suppose that the line is *not* parallel to one of the planes and thus argue to an absurdity.]

13. Prove that if a line is parallel to each of two intersecting planes it is parallel to their intersection.

14. Prove that if a line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

[HINT. Pass a plane through the given line perpendicular to the given plane and use § 267.]

15. Can a trihedral angle be formed by placing three equilateral triangles so that one vertex of each lies at the vertex of the trihedral angle? [HINT. Use § 273.]

16. Can a convex polyhedral angle be formed as in Ex. 15 by placing at its vertex one vertex of each of *four* equilateral triangles? Can this be done with *five* equilateral triangles? With six? With more than six?

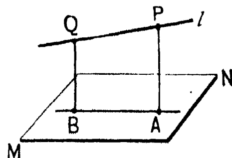
17. Can a convex polyhedral angle be formed by placing at its vertex one vertex of each of three squares? *Four* squares?

18. Can a convex polyhedral angle be formed by placing at its vertex one vertex of each of three regular pentagons? Four?

19. Show that just five different convex polyhedral angles can be formed as in Exs. 15–18 by placing at a single point one vertex of each of several similar regular polygons.

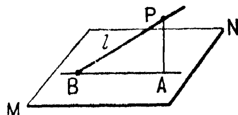
20. Show that the sum of the dihedral angles of a trihedral angle lies between two and six right angles.

21. Is there (in general) a point in space that is equidistant from four given points not all of which lie in the same plane? Give reason for your answer.



22. Given any line l and a plane MN, drop a perpendicular PA from any point P in l to MN. Prove that l and PA determine a plane perpendicular to MN. [This plane is called the **projecting plane** of l on MN. Its intersection AB with MN is called the **projection** of l on MN. Define similarly the projection of the *segment* PQ.]

23. Prove that the **projection** on a plane MN of the line segment joining two points P and Q (Ex. 22) is the line joining the feet A and B of the perpendiculars dropped to the plane from P and Q, respectively.



24. If a line l meets a plane MN at a point B, prove that the projection of l on MN is the line joining B to the foot A of a perpendicular let fall from any point P in l. [The angle ABP between the line l and its projection is called *the angle between the line and the plane*, or the *inclination of the line to the plane.*]

25. Prove that the sides of an isosceles triangle make equal angles with any plane containing its base.

***26.** Show that the length of the projection of any line segment PQ on any plane is the length of PQ times the cosine of the angle between the line and the plane.

CHAPTER VII

POLYHEDRONS CYLINDERS CONES

PART I. PRISMS

274. Polyhedrons. A **polyhedron** is a limited portion of space completely bounded by planes. The intersections of the bounding planes are called the **edges**; the intersections of the edges, the **vertices**; and the portions of the bounding planes bounded by the edges, the **faces**, of the polyhedron.



FIG. 187

A polyhedron of four faces is called a **tetrahedron**; one of six faces (for example, a cube), a **hexahedron**; one of eight faces, an **octahedron**; one of twelve faces, a **dodecahedron**; one of twenty faces, an **icosahedron**.

A **diagonal** of a polyhedron is a straight line joining any two vertices not in the same face.

275. Prisms. A **prism** is a polyhedron, two of whose faces, called its ***bases,*** are congruent polygons in parallel planes, and whose other faces, called ***lateral faces,*** are parallelograms whose vertices all lie in the bases.

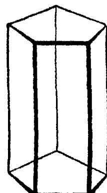
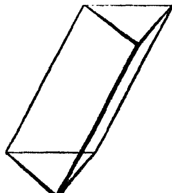
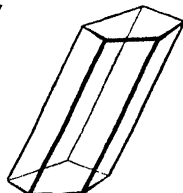
A **triangular** prism is one whose base is a triangle.

The sum of the areas of the lateral faces of any prism is called the **lateral area** of the prism.

The intersections of the lateral faces are the **lateral edges** of the prism.

The **altitude** of a prism is the perpendicular distance between its bases.



RIGHT PRISMS OBLIQUE PRISMS

FIG. 188

A **right prism** is one whose lateral edges are perpendicular to its bases.

An **oblique prism** is one whose lateral edges are oblique to its bases.

A **regular prism** is a right prism whose bases are regular polygons.

Any polygon made by a plane which cuts all the lateral edges of a prism, as the polygon $A'B'C'D'E'$ in Fig. 189, is called a **section** of the prism. A **right section** is one made by a plane perpendicular to all the lateral edges of the prism, as $ABCDE$.

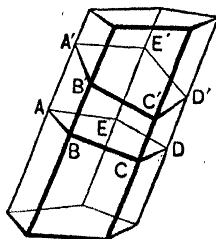


FIG. 189

276. Theorem I. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*

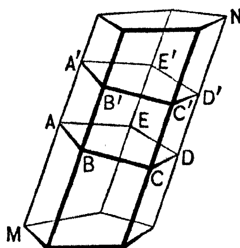


Fig. 190

Given $ABCDE$ and $A'B'C'D'E'$, sections of the prism MN, made by parallel planes.

To prove that $ABCDE \cong A'B'C'D'E'$.

Proof. The sides of the polygon $ABCDE$ are parallel to the sides of the polygon $A'B'C'D'E'$. § 256

Therefore the polygons are mutually equilateral. § 84

Also the polygons are mutually equiangular. § 258

Therefore polygon $ABCDE \cong$ polygon $A'B'C'D'E'$. § 33

EXERCISES

1. How many edges has a tetrahedron? A hexahedron?

2. How many diagonals has a hexahedron? A tetrahedron?

3. Prove that any two lateral edges of a prism are equal and parallel.

4. Prove that any lateral edge of a right prism is equal to the altitude.

5. Prove that all right sections of a prism are congruent.

6. Prove that a section of a prism parallel to the base is congruent to the base.

277. Theorem II. *The lateral area A of a prism is equal to the perimeter* p_r *of a right section multiplied by the lateral edge e; that is,* $A = p_r \times e$.

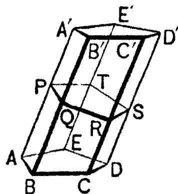


FIG. 191

Given the prism AD' with a right section $PQRST$; let p_r denote the perimeter of the right section, e the lateral edge, and A the lateral area.

To prove that $A = p_r \times e$.

Proof. The lateral area consists of a number of parallelograms, each of which has a line equal to AA' for its base. Why?

Each of these parallelograms has one of the sides of the right section $PQRST$ for an altitude. Why?

Therefore the areas of these parallelograms = the perimeter of $PQRST$ multiplied by AA'. Why?

That is $A = p_r \times e$.

278. Corollary 1. *The lateral area A of a right prism is equal to the perimeter of its base multiplied by a lateral edge;* or $A = p_b \times e$, where p_b denotes the perimeter of the base, and e denotes a lateral edge.

EXERCISES

1. Find the altitude of a regular prism, one side of whose triangular base is 5 in. and whose lateral area is 195 sq. in.

2. Show that the lateral area of a regular hexagonal right prism is $4\sqrt{3} \cdot ah$, where h is the altitude and a the distance from the center of the base to one of the sides.

279. Congruent Solids. Any two solids, in particular any two prisms, are said to be congruent when they can be made to coincide completely in all their parts.

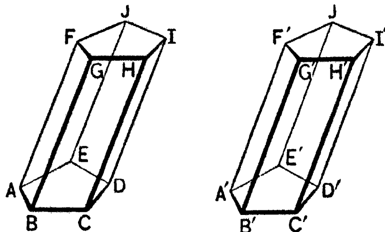


FIG. 192. CONGRUENT PRISMS

280. Theorem III. *Two prisms are congruent if three faces that include a trihedral angle of one are congruent respectively, and similarly placed, to three faces that include a trihedral angle of the other.*

Given the prisms AI and $A'I'$ with face $AJ \cong$ face $A'J'$, face $AG \cong$ face $A'G'$, and face $AD \cong$ face $A'D'$.

To prove that prism $AI \cong$ prism $A'I'$.

Proof. ∠s EAF, FAB, and EAB are equal respectively to ∠s $E'A'F'$, $F'A'B'$, and $E'A'B'$. Why?

Then trihedral $\angle A \cong$ trihedral $\angle A'$. § 271

Place the prism AI on the prism $A'I'$ so that the trihedral $\angle A$ coincides with its congruent trihedral $\angle A'$.

Then the face AJ will coincide with the congruent face $A'J'$; AG with the congruent face $A'G'$; and AD with $A'D'$; and points C and D will fall on C' and D'. § 33

Since the lateral edges of a prism are parallel and equal, CH coincides with $C'H'$, and DI with $D'I'$. §§ 257, 49

Therefore the upper bases coincide, and the prisms coincide throughout and are congruent.

281. Corollary 1. *Two right prisms having congruent bases and equal altitudes are congruent.*

282. Truncated Prisms. A **truncated prism** is a portion of a prism included between the base and a section oblique to the base.

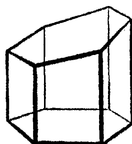


FIG. 193 (*a*)

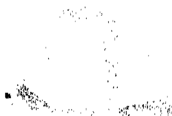


FIG. 193 (*b*)

283. Corollary 2. *Two truncated prisms are congruent if three faces including a trihedral angle of the one are congruent respectively to three faces including a trihedral angle of the other.*

EXERCISES

1. A wooden beam 10 ft. long has a rectangular right cross section whose dimensions are 12 in. by 16 in. If the beam be sawed lengthwise along one of its diagonal planes, show that the resulting triangular prisms are congruent.

2. What will be the lateral area of one of the triangular prisms of Ex. 1? Its *total* area? *Ans.* 40 sq. ft.; $41\frac{1}{3}$ sq. ft.

3. A carpenter is to saw from a given square piece of timber a portion of which one end is to be perpendicular to the lateral edges, while three given lateral edges are to be 3 ft. 6 in., 3 ft. 4 in., and 3 ft. long, respectively. Show that these measurements are sufficient to enable him to saw off the desired portion.

4. Show that to make a right prism of any desired shape, it is sufficient to have a pattern of a right section of the desired prism, and the length of one lateral edge.

5. Show that to make a truncated prism of any desired shape, of which one end is a right section, it is sufficient to have a pattern of that end, and the lengths of three given consecutive lateral edges.

284. Theorem IV. *An oblique prism is equal in volume to a right prism whose base is a right section of the oblique prism and whose altitude is a lateral edge of the oblique prism.*

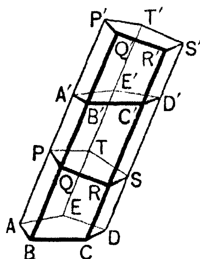


Fig. 194

Given the oblique prism AD'; and given a right prism PS' whose base PS is a right section of the prism AD', and whose altitude is equal to a lateral edge AA' of the prism AD'.

To prove that prism $AD' =$ prism PS'.

Proof. The lateral edges of the prism PS' equal the lateral edges of the prism AD'. Const.

Therefore $AP = A'P'$, $BQ = B'Q'$, etc. Why?

Moreover $PQ = P'Q'$, and the face angles at P, Q, P', Q' are right angles. Why?

Hence, by superposition,

$$\text{Face } AQ \cong \text{Face } A'Q'.$$

Likewise, Face $BR \cong$ Face $B'R'$, etc.

Now, Section $PQRST \cong$ Section $P'Q'R'S'T'$. § 276

Whence, truncated prism $AR \cong$ truncated prism $A'R'$. § 283

Therefore, truncated prism AR + truncated prism PD' = truncated prism $A'R'$ + truncated prism PD'. Ax. 1

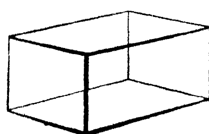
It follows that prism AD' = prism PS'.

285. Equivalent Solids. Two solids that have the same volume are said to be **equivalent, or equal in volume.**

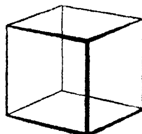
Thus we proved, in § 284, that any oblique prism is *equivalent* to a right prism whose base is a right section of the oblique prism and whose altitude is equal to the lateral edge of the oblique prism.

286. Parallelepipeds. A **parallelepiped** is a prism whose bases are parallelograms.

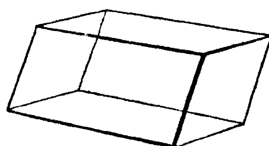
A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to its bases.



RECTANGULAR PARALLELEPIPED



CUBE



OBLIQUE PARALLELEPIPED

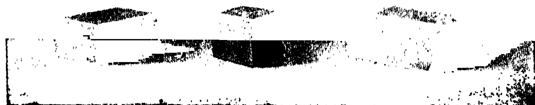


FIG. 195

A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.

A **cube** is a parallelepiped whose six faces are squares.

An **oblique parallelepiped** is one whose lateral edges are oblique to its bases.

EXERCISES

1. Show that the lateral faces of a right parallelepiped are rectangles.

2. Find the sum of all the face angles of a parallelepiped.

3. Find the diagonal of a cube whose edge is 4 in.; 20 in.; a.

287. Theorem V. *The plane passed through two diagonally opposite edges of a parallelepiped divides the parallelepiped into two triangular prisms that are equal in volume.*

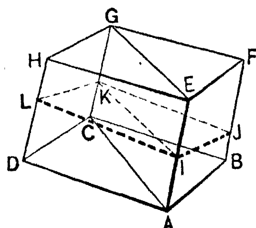


Fig. 196

Given the plane $ACGE$ passing through the opposite edges AE and CG of the parallelepiped AG.

To prove that the parallelepiped AG is divided into two equal triangular prisms, $ABC\text{–}F$ and $ADC\text{–}H$.

Proof. Let $IJKL$ be a right section of the parallelepiped.

From the definition of parallelepiped, the opposite faces, AF and DG, and AH and BG, are parallel and congruent. §§ 258, 88

Therefore $IJ \parallel LK$, and $IL \parallel JK$. § 256

Then $IJKL$ is a parallelogram. Why?

The intersection IK of the right section with the plane $ACGE$ is the diagonal of the $\square$ $IJKL$.

Therefore $\triangle IJK \cong \triangle IKL$. Why?

But the prism $ABC\text{–}F$ is equal in volume to the right prism whose base is IJK and altitude AE, and the prism $ACD\text{–}H$ is equal in volume to the right prism whose base is ILK and altitude AE. § 284

But since these right prisms are congruent, § 281

it follows that prism $ABC\text{–}F =$ prism $ADC\text{–}H$.

Note. If the faces $EFGH$ and $ABCD$ (Fig. 196) are perpendicular to the edges AE, BF, etc., it is easy to see that the diagonal plane $AECG$ divides the parallelepiped into two *congruent* triangular prisms. This happens for any rectangular parallelepiped.

EXERCISES

1. Prove that the diagonals of a rectangular parallelepiped are equal, and that the square of the diagonal is equal to the sum of the squares of the three edges that meet in any vertex.

[HINT. The diagonal is the hypotenuse of a right triangle whose sides are one of the edges and a diagonal of one face; the diagonal of the face is the hypotenuse of a right triangle whose sides are two of the edges.]

2. Find the length of the diagonal of a rectangular parallelepiped whose edges are 8, 10, 12.

3. Find the edge of a cube whose diagonal is 64 in.

4. Prove that the diagonals of a parallelepiped bisect each other.

[HINT. Consider each pair of diagonals separately, and apply § 87 to the diagonal plane in which they lie.]

5. Prove that if the right section *IJKL* (Fig. 196) of a parallelepiped is a rectangle, the two diagonal planes *ACGE* and *BDHF* divide the parallelepiped into four triangular prisms that are equal in volume.

6. A tank in the form of a rectangular parallelepiped that holds 100 gal. is divided into four compartments by two vertical diagonal planes. What is the capacity of each compartment?

7. A cube each of whose edges is 1 ft. long is called a *unit* cube; its volume is one cubic foot. If six such cubes are placed side by side in two rows of three each, they form a rectangular parallelepiped 2 ft. wide, 1 ft. high, and 3 ft. long. What is the volume of this parallelepiped? What is the volume of each of the triangular prisms into which it is divided by a diagonal plane?

8. How many unit cubes are there in a cube each of whose edges is 5 units long?

9. How many unit cubes are there in a rectangular parallelepiped 3 units long, 4 units wide, and 2 units high? What is the volume of this parallelepiped?

288. Volume of a Rectangular Parallelepiped. The three edges of a rectangular parallelepiped which meet at a common point are called its **dimensions.**

In Chapter IV (§ 181), we assumed (without proof) the well-known principle that the area of a rectangle is equal to the product of its two dimensions. Similarly, we shall now assume that the volume of a rectangular parallelepiped is equal to the product of its *three* dimensions, that is, to the product of its length, breadth, and height; *i.e.*

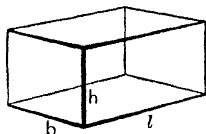


FIG. 197

For any ***rectangular*** *parallelepiped the volume* V *is*

$$V = l \times b \times h,$$

where l, b, h *denote the length, breadth, and height of the parallelepiped.*

The student is reminded that the meaning of the principle is that, if the three dimensions are each measured in terms of the *same* unit of length, then the volume in terms of the *corresponding unit cube* is the product of the three dimensions.

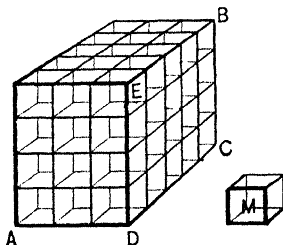


FIG. 198

The following corollaries result at once from this principle:

289. Corollary 1. *Two rectangular parallelepipeds having congruent bases are to each other as their altitudes.*

[HINT. If l, b, and h represent the dimensions of the one parallelepiped, then l, b, and h' will represent the dimensions of the other. The corresponding volumes will therefore be to each other in the ratio $(lbh)/(lbh')$, that is, in the ratio h/h'.]

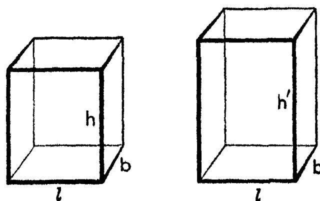


FIG. 199

290. Corollary 2. *Two rectangular parallelepipeds having equal altitudes are to each other as their bases.*

291. Corollary 3. *The volume of a cube is equal to the cube of its edge.*

292. Corollary 4. *The volume V of any* **rectangular** *parallelepiped is the product of the area of its base B and its altitude h; that is, $V = B \times h$.*

EXERCISES

1. Two rectangular parallelepipeds with equal altitudes have bases containing 10 sq. in. and 15 sq. in., respectively. The volume of the first is 56 cu. ft. Find the volume of the second. *Ans.* 84 cu. ft.

2. Compare the volume of the rectangular parallelepiped whose dimensions are 8 in., 10 in., 11 in. with the one whose dimensions are 1 ft., 1 ft., and 16 in.

3. In a lot 120 ft. long and 66 ft. wide a cellar is to be dug for a building. The cellar is to be 44 ft. long, 36 ft. wide, and 7 ft. deep. The earth removed is to be used to fill the surrounding yard. What depth of fill can be made?

4. A **standard (U.S.) gallon** contains 231 cu. in. How many gallons can be put in a tank of the form of a rectangular parallelepiped that is 2 ft. high, $1\frac{1}{2}$ ft. wide, and 3 ft. long?

5. How many gallons (see Ex. 4) are there in 1 cu. ft.?

6. Find the size of a cubical tank that will contain 50 gal.

7. Find the edge of a cube whose volume is 1728 cu. in.; of a cube whose volume is 1500 cu. in.

8. Find the diagonal of a cube whose volume is 521 cu. in.

9. If the volume of one cube is twice that of another, how do their edges compare? *Ans.* $\sqrt[3]{2} : 1$.

10. Find the edge of a cube whose total surface is 60 sq. ft.

11. The edge of a cube is a. Find the area of a section made by a plane through two diagonally opposite edges.

293. Theorem VI. ***The volume V of any parallelepiped is equal to the product of its base B and its altitude h; that is, $V = B \times h$.***

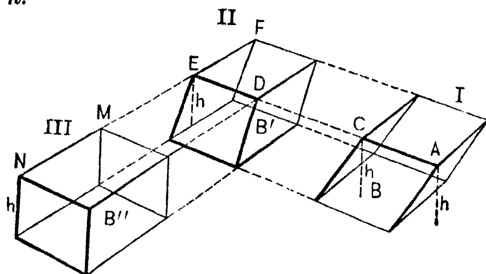


FIG. 200

Given the parallelepiped *I* with its volume denoted by *V*, its base by *B*, and its altitude by *h*.

To prove that $V = B \times h$.

Proof. Produce *AC* and all the other edges of *I* that are parallel to *AC*.

On the prolongation of *AC* take $DE = AC$, and through *D* and *E* pass planes perpendicular to *AE*, forming the right parallelepiped *II* whose base is *B′*.

Then $I = II$. § 284

Prolong *FE* and all the other edges of *II* that are parallel to *FE*.

On the prolongation of *FE* take $MN = FE$, and through *M* and *N* pass planes perpendicular to *FN*, forming the rectangular parallelepiped *III* whose base is *B″*.

Then $II = III$. Why?

Therefore $I = III$. Why?

Moreover $B = B' = B''$; Why?

and *h*, the altitude of *I*, is equal to the altitude of *III*. Why?

But the volume of *III* is $B'' \times h$, by § 288; hence the volume of *I* is $V = B'' \times h = B \times h$.

294. Theorem VII. ***The volume V of any triangular prism is equal to the product of its base*** B ***and its altitude*** h; ***that is,*** $V = B \times h$.

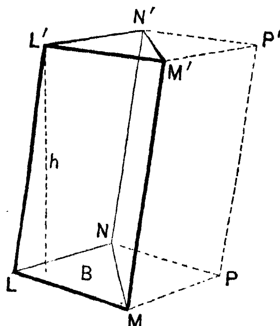


FIG. 201

Given the triangular prism $LMN\text{-}N'$ whose base B is the triangle LMN, and whose altitude is h.

To prove that the volume V of $LMN\text{-}N' = B \times h$.

Proof. Complete the parallelepiped $LMPN\text{-}P'$.

[The remainder of the proof is left to the student. Use § 293.]

295. Corollary 1. *The volume* V *of any prism is equal to the product of its base* B *and its altitude* h; *that is,* $V = B \times h$.

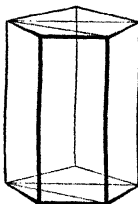


FIG. 202

[HINT. Any prism may be divided into triangular prisms by diagonal planes.]

296. Corollary 2. *Prisms having equivalent bases and equal altitudes are equal.*

EXERCISES

1. Describe one or more ways in which a parallelepiped may be distorted and yet have its volume remain unchanged.

2. The base of a parallelepiped is a rhombus one of whose diagonals is equal to its side. The altitude of the parallelepiped is a, and is also equal to a side of the base. Find the volume of the parallelepiped. *Ans.* $a^3\sqrt{3}/2$.

3. The altitude of a parallelepiped is 3 in., and a diagonal of a base divides the base into two equilateral triangles, each side of which is 6 in. Find the volume of the parallelepiped.

4. The volume of a rectangular parallelepiped is 2430 cu. in. and its edges are in the ratio of 3, 5, and 6. Find its edges.

5. The altitude of a prism is 6 in. and its base is a square each side of which is 3 in. Find its volume.

6. Show that two prisms with equal bases are to each other as their altitudes; and that those with equal altitudes are to each other as their bases.

7. A clay cube having a 2-in. edge is molded into the form of a triangular prism of height 3 in. What is the area of its base? Does it make a difference in the answer whether the prism is made right or oblique? Explain.

8. Assuming that iron weighs about 450 lb. per cu. ft., find the weight of a rod 3 ft. long, whose cross section is a rectangle $1\frac{1}{2}$ by 2 in.

9. With the data of Ex. 8, find the weight of an iron rod 2 ft. 6 in. long, whose cross section is a regular hexagon 1 in. on each side.

10. What must be the length of the side of an equilateral triangle in order that a triangular prism erected upon it and of height 1 ft. shall have a volume of 1 cu. ft.? Solve the same problem, when it is assumed that the base is a regular hexagon.

PART II. PYRAMIDS

297. Pyramids. A **pyramid** is a polyhedron bounded by a polygon, called its **base**, and several triangles that have a common vertex.

The triangles are called the **lateral faces**, the common vertex is called the **vertex** of the pyramid, and the perpendicular distance from the vertex to the base is called the **altitude**.

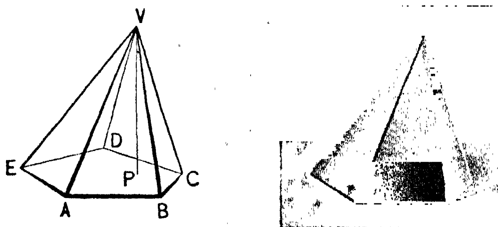


FIG. 203. PYRAMID

A pyramid is *triangular, quadrangular,* etc., according as its base is a triangle, a quadrilateral, etc.

A **regular pyramid** is one whose base is a regular polygon and whose vertex lies in the perpendicular erected at the center of the base.

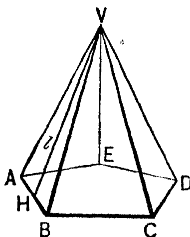


FIG. 204. REGULAR PYRAMID

The **slant height** of a *regular* pyramid is the altitude of any one of its triangular faces (*VH* in Fig. 204).

A **truncated pyramid** is the portion of a pyramid included between the base and any section made by a plane cutting all the lateral edges.

A **frustum of a pyramid** is a portion included between the base and a section made by a plane parallel to the base.

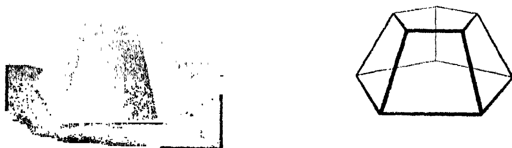


FIG. 205. FRUSTUM OF A PYRAMID

The *altitude of a frustum* is the length of the perpendicular between the planes of its bases.

The *lateral faces* of a frustum of a *regular* pyramid are congruent trapezoids.

The *slant height* of the frustum of a *regular* pyramid is the altitude of one of the trapezoids forming its faces.

EXERCISES

1. Of which type are the celebrated Egyptian pyramids?

2. Prove the equality of the lateral edges of a regular pyramid. Of those of a frustum of a regular pyramid.

3. Prove that the faces of any frustum of a pyramid are trapezoids.

4. Prove the statement made in § 297 that the faces of a frustum of a *regular* pyramid are *congruent* trapezoids.

5. Prove that the lateral faces of a regular pyramid are congruent isosceles triangles; hence show that the slant height, measured on any face, is the same as that measured on any other face.

6. Prove that any triangular pyramid is a tetrahedron. State and prove the converse.

298. Theorem VIII. ***The lateral area A of a regular pyramid is equal to one half the product of the perimeter of its base p, and its slant height l; that is, $A = p \times l/2$.***

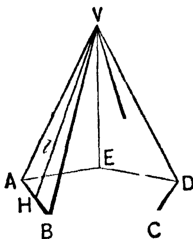


FIG. 206

Given the regular pyramid $V\text{-}ABCDE$ with the slant height l, the lateral area A, and the perimeter of the base p.

To prove that $A = p \times l/2$.

Proof. $A =$ the sum of the areas of the triangles VAB, VBC, etc.

Hence $A = [AB + BC + \cdots] \times l/2 = p \times l/2$. Why?

299. Corollary 1. *The lateral area of the frustum of a regular pyramid is equal to one half the product of the sum of the perimeters of the bases and its slant height.* [See § 191.]

EXERCISES

1. The slant height of a regular hexagonal pyramid is 10 ft. Each side of its base is 8 ft. What is its lateral area? Also, what is its total area? *Ans.* 240 sq. ft.; 406.27 sq. ft.

2. The altitude of a regular quadrangular pyramid is 4 in. One side of its base is 6 in. What is its lateral area? What is its total area? *Ans.* 60 sq. in.; 96 sq. in.

3. Find the lateral area of the frustum formed by a plane bisecting the altitude of the pyramid mentioned in Ex. 2. Find its total area.

300. Theorem IX. *If a pyramid is cut by a plane parallel to the base,*

(a) The altitude and the lateral edges are divided proportionally;

(b) The section is a polygon similar to the base.

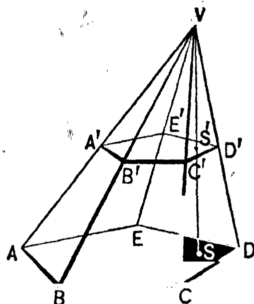


Fig. 207

Given the pyramid $V\text{–}ABCDE$ cut by a plane $A'D'$ parallel to the base AD.

To prove (*a*) that

$$VS/VS' = VA/VA' = VB/VB' = \cdots, \text{ etc.}$$

(*b*) that the section $A'B'C'D'E'$ is similar to the base.

Proof of (*a*)

Pass a plane through V parallel to the base and apply § 261.

Proof of (*b*)

$\triangle VAB \sim \triangle VA'B'$; $\triangle VBC \sim \triangle VB'C'$, etc. Why?

Therefore

$AB/A'B' = VB/VB'$; $VB/VB' = BC/B'C'$, etc.; Why?

and hence $AB/A'B' = BC/B'C' = CD/C'D' = \cdots$ Ax. 9

Thus, the polygons $ABCDE$ and $A'B'C'D'E'$ have their corresponding sides proportional.

Moreover, the same polygons are mutually equiangular. § 258

Hence $ABCDE \sim A'B'C'D'E'$. § 165

301. Corollary 1. *Parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

Proof: In Fig. 207, $ABCDE/A'B'C'D'E' = \overline{AB}^2/\overline{A'B'}^2$. §§ 195, 300

But $AB/A'B' = VB/VB'$, Why?

and also $VS/VS' = VB/VB'$; (*a*), § 300

hence $AB/A'B' = VS/VS'$. Ax. 9

Whence, squaring, $\overline{AB}^2/\overline{A'B'}^2 = \overline{VS}^2/\overline{VS'}^2$.

Hence $ABCDE/A'B'C'D'E' = \overline{VS}^2/\overline{VS'}^2$.

302. Corollary 2. *If two pyramids that have equivalent bases and equal altitudes are cut by planes parallel to their bases and at equal distances from their vertices, the sections are equivalent.*

[Hint. Represent by R and R' the areas of the two sections, and by B and B' the areas of the bases. Let h be the common altitude of the pyramids, and k the distance from the vertex of either pyramid to the section made in it. Then $R/B = k^2/h^2$ and $R'/B' = k^2/h^2$ (§ 301); hence $R/B = R'/B'$. But $B = B'$ by hypothesis; hence $R = R'$.]

EXERCISES

1. Compare the areas of two sections of a pyramid whose perpendicular distances from the vertex are 3 in. and 4 in. respectively. Does it make any difference in your answer whether the pyramid is of one shape or another? *Ans.* 9 : 16.

2. The altitude of a pyramid with a square base is 16 in., the area of a section parallel to the base and 10 in. from the vertex is $56\frac{1}{4}$ sq. in. Find the area of the base.

3. The bases of the frustum of a regular pyramid are equilateral triangles whose sides are 10 in. and 18 in. respectively; the altitude of the frustum is 8 in. Find the altitude of the pyramid of which the given figure is a frustum. *Ans.* 18 in.

4. The altitude of a pyramid is H. At what distance from the vertex must a plane be passed parallel to the base so that the section formed shall be (1) one half as large as the base? (2) one third? (3) one ninth?

303. Theorem X. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*

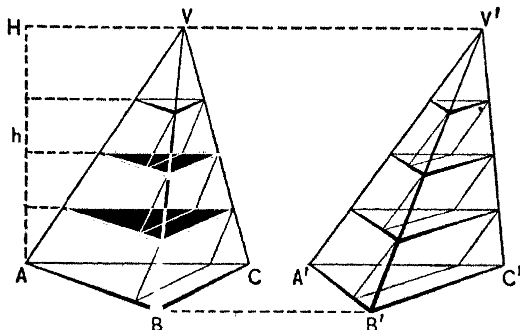


FIG. 208

Given the pyramids V–ABC and V'–$A'B'C'$ having equivalent bases ABC, $A'B'C'$, and a common altitude AH.

To prove $V\text{–}ABC = V'\text{–}A'B'C'$.

Proof. The proof of the theorem consists in showing that the pyramids V–ABC and V'–$A'B'C'$ cannot differ in volume by as much as any given amount, however small. This means that the two volumes are actually equal, for if they were unequal, they would differ by as much as some *fixed* amount, — in fact, that is what unequal means.

We proceed, then, to show that V–ABC and V'–$A'B'C'$ cannot differ by as much as any given amount, however small.

Divide the altitude AH into a number of equal parts, and through the points of division pass planes parallel to the plane of the bases.

The corresponding sections made by any one of these planes in the two pyramids are equivalent. § 302

Now inscribe in each pyramid a series of prisms having the triangular sections as upper bases and the distance between the sections as their common altitude.

Each pair of corresponding prisms in the two pyramids are then equivalent. § 296

Therefore, the sum of the prisms inscribed in V–ABC is equivalent to the sum of the prisms inscribed in V'–$A'B'C'$.

If the number of divisions into which AH is divided is taken sufficiently large, the sum of the prisms in V–ABC may be made to differ from the volume of V–ABC by less than any given amount. Likewise, by taking the number of divisions in AH sufficiently large, the sum of the prisms in V'–$A'B'C'$ may be made to differ from the volume of V'–$A'B'C'$ by less than the same given amount, however small it has been taken.

Since, by taking the number of divisions in AH sufficiently large, the volumes of V–ABC and V'–$A'B'C'$ differ by less than any given amount from these *equal* sums, the pyramids must differ from each other by less than the same given amount, and this is what we were to show. Compare § 211.

304. Corollary 1. *Any two pyramids having equivalent bases and equal altitudes are equivalent.*

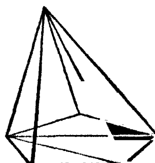


Fig. 209

[Hint. Divide each pyramid into triangular pyramids.]

EXERCISES

1. What is the locus of the vertices of all pyramids whose bases and volumes are the same?

2. Prove that if the base of a pyramid is a parallelogram, the plane determined by its vertex and either diagonal of its base divides it into two equivalent triangular pyramids.

305. Theorem XI. ***The volume V of a triangular pyramid is equal to one third the product of its base B, and its altitude h; that is,*** $V = B \times h/3$.

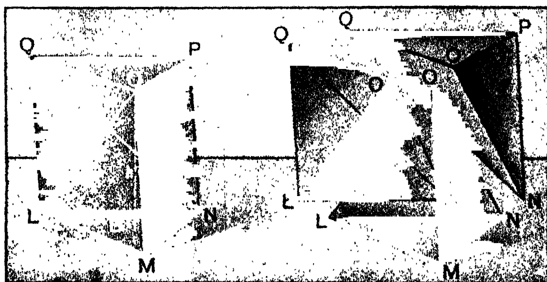


FIG. 210

Given the triangular pyramid *O–LMN.*

To prove that the volume *V* of *O–LMN* equals $\frac{1}{3}$ its base *B* times its altitude *h*; that is, $V = B \times h/3$.

Proof. Construct a triangular prism *MP* having *LMN* for its base, and its lateral edges equal and parallel to the edge *OM.*

The prism *MP* is made up of the triangular pyramid *O–LMN* and the quadrangular pyramid *O–LNPQ.*

Pass a plane through *OQ* and *ON* dividing the quadrangular pyramid into two triangular pyramids, *O–LNQ* and *O–NQP.*

Pyramid *O–LNQ* = pyramid *O–NQP.* § 303

Pyramid *O–NQP* may be read *N–QOP.*

Pyramid *N–QOP* = pyramid *O–LMN.* § 303

Therefore, the three triangular pyramids are equal and *O–LMN* is one third the prism.

But the volume of the prism is equal to the product of its base and its altitude. § 294

Therefore, pyramid *O–LMN* = $\frac{1}{3}$ the product of its base and its altitude.

306. Corollary 1. *The volume V of any pyramid is equal to one third the product of its base B and its altitude h; that is,* $V = Bh/3$.

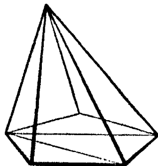


FIG. 211

[HINT. Divide the pyramid into triangular pyramids and apply § 305.]

307. Theorem XII. *If a frustum of a pyramid has bases B and B' and altitude h, and is cut from a pyramid P whose base is B and whose altitude is H, the volume V of the frustum is given by the formula:*

$$V = \frac{BH}{3} - \frac{B'(H-h)}{3}.$$

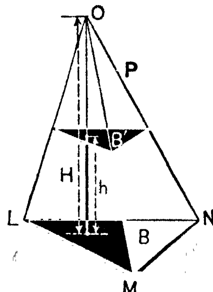


FIG. 212

Given the pyramid P with base B and altitude H, and a frustum of it with bases B and B' and altitude h.

To prove that the volume V of the frustum is

$$V = \frac{BH}{3} - \frac{B'(H-h)}{3}.$$

Proof. The frustum is the difference between the two pyramids P and P', where P' has the base B' and the same vertex as P.

The volume of P is $BH/3$. Why?

Since the altitude of P' is $H - h$, its volume is

$$\frac{B'(H-h)}{3}. \qquad \text{Why?}$$

Hence $$V = P - P' = \frac{BH}{3} - \frac{B'(H-h)}{3}.$$

308. Corollary 1. *The volume V of a frustum of a pyramid of bases B and B′ and altitude h is given by the formula:*

$$V = (B + B' + \sqrt{BB'})h/3.$$

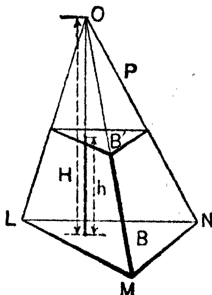


FIG. 213

Outline of Proof. By § 301,

$$B'/B = (H - h)^2/H^2,$$

using the notation of § 307.

Hence $\sqrt{B'}/\sqrt{B} = (H - h)/H = 1 - h/H,$

whence $$H = h\sqrt{B}/(\sqrt{B} - \sqrt{B'}).$$

But, by § 307,

$$V = BH/3 - B'(H - h)/3$$
$$= (B - B')H/3 + B'h/3.$$

Substituting the value of H just found, we have

$$V = \left[\frac{(B - B')\sqrt{B}}{\sqrt{B} - \sqrt{B'}} + B'\right]h/3 = [B + \sqrt{BB'} + B']h/3.$$

EXERCISES

1. Find the altitude of a triangular pyramid whose volume is 50 cu. in. and whose base is 12 sq. in. *Ans.* $12\frac{1}{2}$ in.

2. If a prism and a pyramid have a common base and altitude, what is the ratio of their volumes?

3. If the base of a pyramid is a square and its altitude is 3 ft., how long must each side of the square be in order that the volume may be 16 cu. ft.?

4. Show that the volume of the tetrahedron, all of whose edges are equal to a, is $\sqrt{2}\,a^3/12$.

[HINT. See Th. XXXIII, § 102.]

5. Find the volume of a frustum of the pyramid of Ex. 1 cut off by a plane 6 in. from the base.

6. Find the volume of each of the parts into which the pyramid of Ex. 3 is cut by two planes parallel to its base which trisect the altitude.

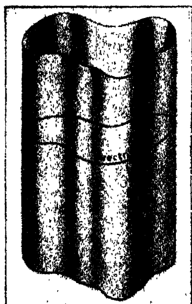
PART III. CYLINDERS AND CONES

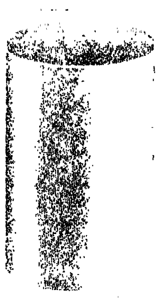
309. Cylinders. A **cylindrical surface** is a curved surface generated by a moving straight line, called the **generatrix**, which moves always parallel to itself and constantly passes through a fixed curve called the **directrix**. The generatrix in any one position is called an **element** of the surface. One element and only one can be drawn through a given point on the cylindrical surface.

A **cylinder** is a solid bounded by a cylindrical surface and two parallel planes. The two plane surfaces are called the **bases**, and the cylindrical surface is called the **lateral surface**.

The **altitude** of a cylinder is the length of the perpendicular between the bases.

A **right section** of a cylinder is a section made by a plane perpendicular to all its elements.


CYLINDRICAL SURFACE


RIGHT CYLINDER


OBLIQUE CYLINDER

FIG. 214

A **circular cylinder** is one whose bases are circles.

A **right cylinder** is one whose elements are all perpendicular to its bases; otherwise, the cylinder is said to be *oblique*.

A **right circular cylinder** is a right cylinder whose base is a circle. Such a cylinder can be generated by the revolution of

a rectangle about one of its sides as an axis; for this reason a right circular cylinder is sometimes called a *cylinder of revolution.*

310. Postulate. A prism is **inscribed** in a cylinder when its lateral edges are elements of the cylinder and its bases are inscribed in the bases of the cylinder.

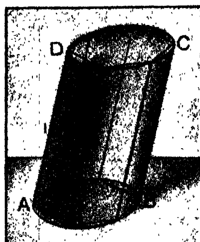


Fig. 215

In order to study the properties of the cylinder the following postulate is needed:

A cylinder and a prism inscribed within it may be made to differ by as little as we please, both in lateral area and in volume, by making the number of sides of the base of the prism sufficiently great, while the length of each of those sides becomes sufficiently small.

The length of an edge of the inscribed prism is equal to the length of an element of the cylinder (see Ex. 5, p. 235); and, by increasing the number of sides of the inscribed prism, the base of the prism approaches, both in area and in the length of its perimeter, as nearly as we please to the base of the cylinder. This latter fact we assume, as in § 210.

We shall now proceed to show that the theorems already proved for prisms can be extended to cylinders by the use of the preceding postulate.

311. Theorem XIII. *The lateral area* A *of any cylinder is equal to the product of an element* l *and the perimeter* p *of a right section; that is,* $A = l \times p$.

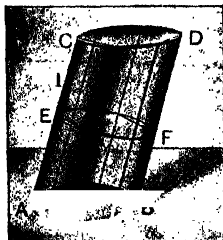


FIG. 216

Outline of Proof. In order to prove the theorem, inscribe in the cylinder any prism whose base is a polygon of n sides. Then, for this *prism*, by § 277:

(1) $$A' = l \times p',$$

where A' and p' are, respectively, the lateral area and the perimeter of the right section of the prism; and where l is the length of an edge of the prism, which is equal to an element of the cylinder. As the number n of sides increases so that the length of each side approaches zero,

A' comes to differ from A by as little as we please; § 310.

$l \times p'$ comes to differ from $l \times p$ by as little as we please. § 310.

Hence, by (1), A comes to differ from $l \times p$ by as little as we please.

It follows, as in § 303, that $A = l \times p$.

312. Corollary 1. *The lateral area of a right circular cylinder is equal to* $2\pi rh$, *where* r *is the radius of the circular base and* h *is the altitude of the cylinder.* See § 214.

313. Corollary 2. *The lateral area of any cylinder whose right section is a circle is equal to* $2\pi rl$, *where* r *is the radius of the right section, and* l *is the length of an element.*

314. Theorem XIV. *The volume V of any cylinder is equal to the product of its base B and its altitude h; that is, $V = B \times h$.*

[The proof is left to the student. Inscribe a prism of n sides in the cylinder and use § 295 and § 310. Follow the steps suggested by § 311.]

315. Corollary 1. *The volume of a circular cylinder is equal to $\pi r^2 h$, where r is the radius of the base and h is the altitude of the cylinder.* See § 216.

EXERCISES

[In these exercises, use the approximate value $\pi = 22/7$.]

1. In a steam engine 65 flues, or cylindrical pipes, each 2 in. in outside diameter and 12 ft. long, convey the heat from the fire-box through to the water. How much heating surface is presented to the water? *Ans.* $408\frac{4}{7}$ sq. ft.

2. Neglecting the lap, how much tin is required to make a stovepipe 10 ft. long and 8 in. in diameter?

3. A right circular cylinder has the radius of its base equal to 3 in. How great must its altitude be in order that it shall have a lateral area of 30 sq. in.?

4. Find the total area, including the ends, of a covered tin can whose diameter is 4 in. and whose height is 6 in. *Ans.* $100\frac{4}{7}$ sq. in.

5. Derive a general formula for the total area (including the bases) of a right circular cylinder whose height is h and the radius of whose base is r.

6. What fraction of the metal in a tin can 5 in. wide and 5 in. high is used to make the top and bottom? What to make the circular sides?

7. If the diameter of a well is 7 ft. and the water is 10 ft. deep, how many gallons of water are there in it, reckoning $7\frac{1}{2}$ gal. to the cubic foot? *Ans.* 2887.5 gals.

8. When a body is placed in a cylindrical tumbler of water 3 in. in diameter, the water level rises 1 in. What is the volume of the body? Note that a method for finding the volume of a body of *any* shape is here illustrated.

9. Show that the volume V and the lateral area A of a right circular cylinder are connected by the relation $V = A \times r/2$.

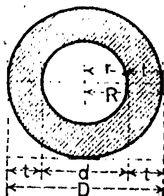
10. One gallon is 231 cubic inches. At what heights on a cylindrical measuring can whose base is 6 in. in diameter will the marks for 1 gallon, 1 quart, 2 quarts, 3 quarts, be made?

11. Find the total area of the gallon measuring can of Ex. 10.

12. Having given the total surface T of a right circular cylinder, in which the height is equal to the diameter of the base, find the volume V.

[HINT. — Call the height h.]

13. Find the volume of metal per foot of length in a pipe whose outer diameter is $3\frac{1}{2}$ in., and whose inner diameter is 3 in. Hence find the weight per foot of length if the pipe is iron, it being given that iron weighs (about) 450 lb. per cubic foot.



14. If the outer and inner diameters of a tube are D and d, respectively, show that the volume in a length l is $\pi l(D^2 - d^2)/4$. If the thickness of the tube is denoted by t, show that $t = (D - d)/2$, and hence that the volume in a length l is

$$\pi lt(D + d)/2.$$

15. Show that the volume of the tube of Ex. 14 can also be represented by the formula $\pi lt(d + t)$; or by the formula $\pi lt(D - t)$, in the notation of Ex. 14.

16. What is the volume of the largest beam of square cross section that can be cut from a circular log 16 in. in diameter and 10 ft. long? What fraction of the log is wasted?

316. Cones. A **conical surface** is a surface generated by a moving straight line *AB*, called the **generatrix**, which passes always through a fixed point *V*, called the **vertex**, and constantly



FIG. 217

intersects a fixed curve, called the **directrix**. A conical surface thus consists of two parts, which are called the upper and lower **nappes**. The generatrix in any one position is called an **element** of the surface.

A **cone** is a solid bounded by a conical surface and a plane which cuts all its elements. The plane is then called the **base** of the cone; and the conical surface is called the **lateral surface** of the cone. The **altitude** of a cone is the perpendicular distance from its vertex to its base.

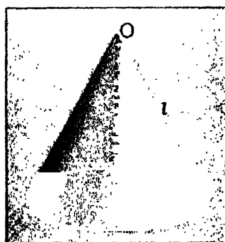


FIG. 218

A **circular** cone is one whose base is a circle. The **axis** of a circular cone is the line joining the vertex to the center of the base.

A **right circular cone** is a circular cone whose axis is perpendicular to the base. Such a cone is also called a **cone of revolution**, since it may be generated by revolving a right triangle about one of its sides as an axis.

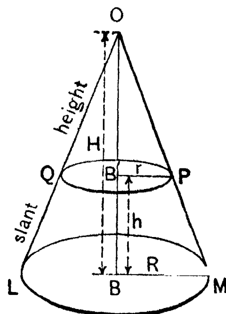


FIG. 219. RIGHT CIRCULAR CONE AND SECTION PARALLEL TO BASE

The **slant height** of a cone of revolution is the length of one of its elements.

A **frustum** of a cone is the portion of a cone included between the base and a section parallel to the base and cutting all the elements.

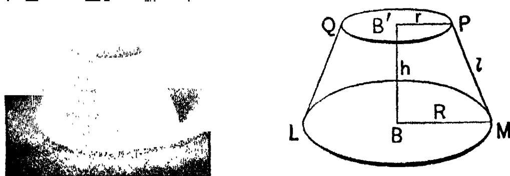


FIG. 220. FRUSTUM OF A CONE

The **lateral surface** of a frustum of a cone is the portion of the lateral surface of the cone included between the bases of the frustum.

The **slant height** of a frustum of a cone of revolution is the portion of any element of the cone included between the bases.

317. Postulate. A pyramid is **inscribed** in a cone when its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

The following postulate, corresponding to that of § 310, is needed in the study of the cone:

A cone and pyramid inscribed within it may be made to differ by as little as we please, both in lateral area and in volume, by making the number of sides of the pyramid sufficiently great, while the length of each side of the base becomes sufficiently small.

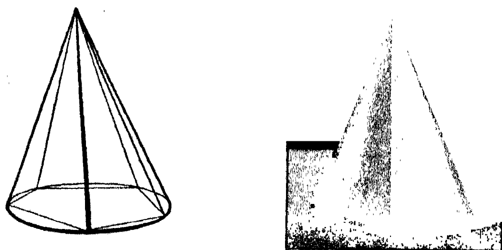


FIG. 221. CONE WITH INSCRIBED PYRAMID

The base of the inscribed pyramid approaches, both in area and in perimeter, the base of the cone (§ 310); and the altitude of any face of the pyramid approaches an element of the cone, as the pyramid approaches the cone.

If the cone is a right circular cone, the pyramid can be made a regular pyramid; then the slant height of the pyramid approaches the slant height of the cone.

318. Restriction. The word *cone* as used hereafter in this book will be understood to refer to a *circular* cone only. The preceding postulate applies, however, to any kind of cone; and it may be used to obtain results for cones of any form in the manner illustrated below for circular cones only.

We proceed to extend to circular cones certain theorems already proved for pyramids.

319. Theorem XV. *The lateral area* A *of a right circular cone is equal to one half the product of its slant height* l *and the circumference* p *of its base; that is,* $A = l \times p/2$.

Outline of Proof. Inscribe a regular pyramid of n faces in the cone (see Fig. 221); then, by § 298, the lateral area A' of the pyramid is equal to one half the product of its slant height l' and the perimeter p' of its base; that is,

$$A' = \tfrac{1}{2}\, l' \times p';$$

and this formula is correct no matter how large n may be.

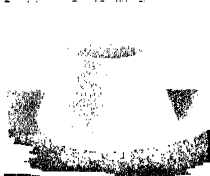
By taking n sufficiently large, A' comes to differ by as little as we please from A; while l' and p' come to differ by as little as we please from l and p, respectively. § 317.

Whence, as in § 311,

$$A = \tfrac{1}{2}\, l \times p.$$

320. Corollary 1. *The lateral area of a right circular cone is* $\pi r \cdot l$, *where* r *is the radius of the base and* l *is the slant height.* See §§ 319 and 214.

321. Corollary 2. *The lateral area of a frustum of a right circular cone is equal to one half the product of its slant height and the sum of the circumferences of its bases.*



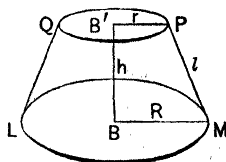


FIG. 222

[The proof is left to the student. Inscribe a frustum of a regular pyramid in the given frustum of a cone, and use § 317 and § 299.]

322. Theorem XVI. *Any section of a circular cone parallel to its base is a circle whose area is to that of the base as the square of its distance from the vertex is to the square of the altitude of the cone.* [HINT. To prove that the section is a circle, pass any two planes through the axis of the cylinder, and show that their intersections with the section are equal. Then inscribe a regular pyramid and proceed as in § 319, using §§ 301 and 317.]

323. Theorem XVII. *The volume V of a cone is equal to one third the product of its base B and its altitude h; that is, $V = Bh/3$.*

[HINT. Use §§ 317, 306, and proceed as in § 319.]

324. Corollary 1. *If from any cone whose base is B and whose altitude is H, a frustum is cut, whose upper base is B' and whose altitude is h, the volume V of the frustum is*

$$V = \frac{BH}{3} - \frac{B'(H-h)}{3}.$$

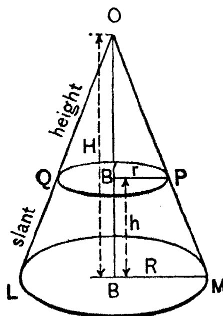


FIG. 223

325. Corollary 2. *The volume of a frustum of any cone is equal to the sum of three cones whose common altitude is the altitude of the frustum and whose bases are the two bases and a mean proportional between them.*

[HINT. Use §§ 322, 324, noting also § 308.]

EXERCISES

1. The altitude of a right circular cone is 12 in. and the radius of the base 9 in. Find the lateral area and the total area of the cone. *Ans.* $424\frac{2}{7}$ sq. in.; $678\frac{6}{7}$ sq. in.

2. How many square yards of canvas are there in a conical tent 12 ft. in diameter and 8 ft. high?

3. The total area of a right circular cone whose altitude is 10 in. is 280 sq. in. Find the total area of the cone cut off by a plane parallel to the base and 6 in. from the vertex.

4. The altitude of a right circular cone is H. How far from the vertex must a plane be passed parallel to the base so that the lateral area of the cone cut off shall be one half that of the original cone? *Ans.* $H/\sqrt{2}$.

[HINT. First prove that the area of the cross section made by the plane will be one half the area of the base. Then apply § 322.]

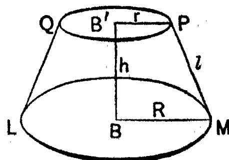
5. The slant height and the diameter of the base of a right circular cone are each equal to l. Find the total area, including the base.

6. The circumference of the base of a circular cone is 11 ft. and its height is 8 ft. What is its volume?

Ans. $\dfrac{242}{3\pi}$, or $25\frac{2}{3}$ cu. ft.

7. If the height of a circular cone is 10 ft., what must be the radius of its base in order that the volume may be 30 cu. ft.?

8. A frustum of a cone is 1 ft. high and the radii of its bases are respectively 9 ft. and 4 ft. Find its volume.



9. If r and R are the radii of the bases of the frustum of a cone and l is its slant height, find the formula for its volume.

PART IV. GENERAL THEOREMS ON POLYHEDRONS
SIMILARITY REGULAR SOLIDS VOLUMES

326. Theorem XVIII. *Two triangular pyramids that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the edges including the equal trihedral angles.*

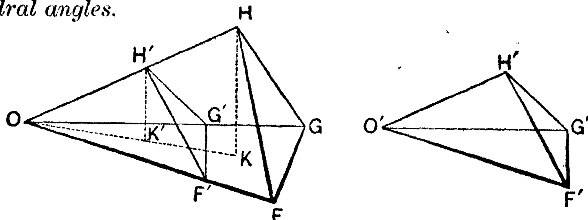


FIG. 224

Given the triangular pyramids $O\text{–}FGH$ and $O'\text{–}F'G'H'$, with the trihedral $\angle O =$ trihedral $\angle O'$, and with volumes denoted by V and V', respectively.

To prove that $V/V' = [OF \cdot OG \cdot OH]/[O'F' \cdot O'G' \cdot O'H']$.

Proof. Place pyramid $O'\text{–}F'G'H'$ so that trihedral $\angle O'$ will coincide with trihedral $\angle O$.

From H and H' draw HK and $H'K'$ perpendicular to the plane OFG.

Then $V/V' = [\triangle OFG \cdot HK]/[\triangle OF'G' \cdot H'K']$

$= [\triangle OFG]/[\triangle OF'G'] \cdot [HK/H'K']$. Why?

But $$\frac{\triangle OFG}{\triangle OF'G'} = \frac{OF \cdot OG}{OF' \cdot OG'}.$$ § 193

Again, let the plane determined by HK and $H'K'$ intersect plane OFG in line $OK'K$.

Then rt. $\triangle OKH \sim$ rt. $\triangle OK'H'$. Why?

Therefore $HK/H'K' = OH/OH'$ Why?

Therefore $$\frac{V}{V'} = \frac{OF \cdot OG}{OF' \cdot OG'} \cdot \frac{OH}{OH'} = \frac{OF \cdot OG \cdot OH}{O'F' \cdot O'G' \cdot O'H'}.$$

327. Corollary 1. *Two triangular prisms that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the edges including the trihedral angles.*

[HINT. Break the prism up into triangular pyramids, and use § 326 and Theorem *II*, § 144.]

328. Corollary 2. *Two parallelepipeds that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the edges including the trihedral angle.*

329. Similar Tetrahedrons. Two tetrahedrons (that is, triangular pyramids) are said to be **similar** if their faces are similar each to each and similarly placed.

330. Theorem XIX. *The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.*

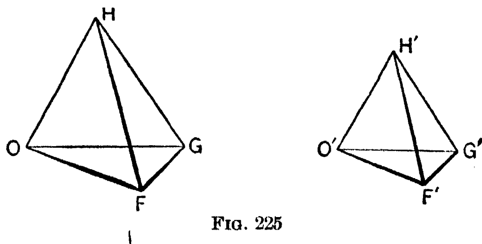


FIG. 225

Given the similar tetrahedrons $O\text{–}FGH$ and $O'\text{–}F'G'H'$ with the volumes denoted by V and V', and with OF and OF' two corresponding edges.

To prove that $V/V' = \overline{OF}^3/\overline{O'F'}^3$. §§ 158, 271

Proof. Trihedral $\angle O =$ trihedral $\angle O'$. § 329

Therefore

$$V/V' = \frac{OF \cdot OG \cdot OH}{O'F' \cdot O'G' \cdot O'H'} = \frac{OF}{O'F'} \cdot \frac{OG}{O'G'} \cdot \frac{OH}{O'H'}. \quad \text{§ 326}$$

But $OF/O'F' = OG/O'G' = OH/O'H'$. Why?

Therefore

$$V/V' = (OF/O'F')(OF/O'F')(OF/O'F') = \overline{OF}^3/\overline{O'F'}^3.$$

331. Similar Polyhedrons. In general, **similar polyhedrons** are polyhedrons which have the same number of faces similar each to each and similarly placed, and their corresponding polyhedral angles equal.

In the case of similar tetrahedrons, the trihedral angles of the one are necessarily equal to those of the other, if we know only that the faces are similar each to each, since the similarity of the faces makes the three face angles at each vertex equal in the two tetrahedrons, by § 158.

By § 330 and Theorem *H*, § 144, we can show that any two similar polyhedrons are to each other as the cubes of any two corresponding edges.

332. The Regular Solids. A **regular polyhedron** is one whose faces are all congruent regular polygons and whose polyhedral angles are all likewise congruent. Five types of such polyhedrons are represented below.



FIG. 226. — THE FIVE REGULAR SOLIDS

We shall now show that the above types are the *only* possible types of regular polyhedrons.

333. Theorem XX. *There exist only five different types of regular polyhedrons.*

Proof. The proof is based upon two facts: (1) that any polyhedral angle has at least three faces and (2) that the sum of the face angles of any convex polyhedral angle must be less than 360°. (See § 273.)

Suppose first that each face is to be a triangle. Then, from the definition of a regular polyhedron, the triangle must be equilateral. Each of its angles will therefore be 60°. Consequently, by statement (2) above, polyhedral angles may be formed by combining three, four, or five such angles, but no more than five can be thus used, since six such angles amount to 360°, while seven or more of them exceed 360°.

Therefore, not more than three regular polyhedrons are possible having triangles as faces. The three that *are* possible are the regular tetrahedron, regular octahedron, and regular icosahedron. (See Fig. 226.)

Suppose secondly that each face is to be a square. Each face angle must then be 90° and, the sum of four such angles being 360°, it follows that but one regular polyhedron is possible having squares as sides. The cube is the one that *is* possible.

Thirdly, suppose that each face is to be a regular pentagon. Since each of the angles of such a figure is 108°, it follows that no more than one regular polyhedron is possible whose faces are pentagons. The one that *is* possible is the dodecahedron.

We can proceed no farther, for the sum of three angles of a regular hexagon is 360°, while the sum of three angles of any regular polygon of more than six sides is greater than 360°.

Hence the theorem is proved.

NOTE. These regular solids occur in nature in the forms of a variety of crystals; but not all crystals are regular solids.

334. Models. Models of the five possible regular polyhedrons can be easily constructed as follows:

Draw diagrams on cardboard as indicated in the figures below. Cut these out and then cut half way through the dotted lines so as to make it easy to fold along these lines.

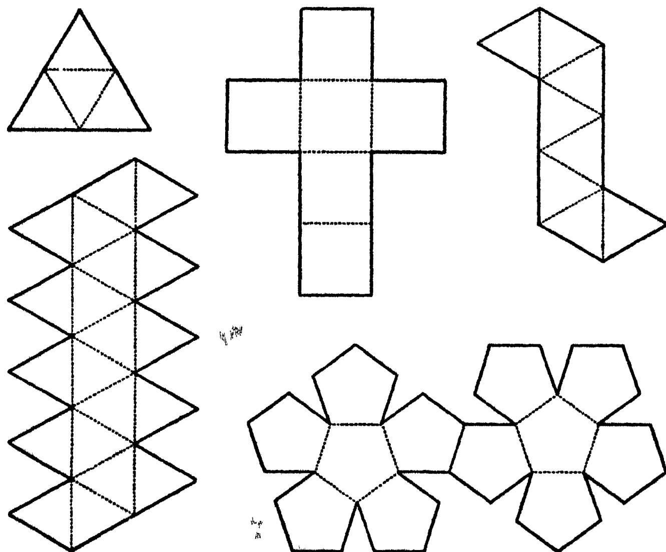


Fig. 227

Fold on the dotted lines so as to bring the edges together, subsequently pasting strips of paper over the edges to hold the solid in position. Models of the tetrahedron, octahedron, and icosahedron may also be made very quickly by hinging together short umbrella wires by means of strong copper wires strung through the end holes, joining together at each corner the proper number of rods. The student may show that each of these models will be quite rigid when completed.

335. Theorem XXI. Cavalieri's Theorem. *If two solids are included between the same pair of parallel planes, and if every section of one of the solids by any plane parallel to one of these parallel planes is equal in area to the section of the other solid by the same plane, the volumes of the two solids are equal.*

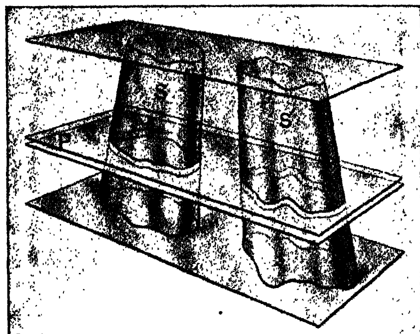


FIG. 228

Outline of Proof. The two solids may be divided into a large number of thin slices by sections parallel to the two including planes. These slices may be thought of as approximately cylindrical, and the sum of all the slices in either case is the volume of the solid.

The bases of two corresponding slices of the two solids between the same two planes are equal in area, by hypothesis. It is therefore apparent that the *volumes* of the two corresponding slices differ as little as we please if their thickness is sufficiently small. It can be shown in a precise manner that the total volume V of one of the solids, which is the sum of all such slices, differs from the total volume V' of the other solid by as little as we please.

Hence, as in §§ 303, 311, it follows that $V = V'$.

NOTE. Observe also that §§ 293, 303 are essentially special cases of what precedes.

336. Theorem XXII. The Prismoid Formula. *If any solid S of any of the kinds considered in this Chapter is bounded by two parallel plane sections B and T, and if M denotes the area of another section parallel to and midway between B and T, the volume V of S is given by the formula:*

$$V = (B + T + 4\,M) \cdot h/6,$$

where B, T, and M denote the areas of the sections, and h denotes the distance between B and T.

The proof of the preceding formula consists in showing that it reduces in every case to the very formula for volume that has already been proved in the articles above.

Outline of Proof for Prisms and Cylinders. In these cases, all parallel sections are equal (§§ 276, 310). Hence $B = T = M$; and the formula to be proved becomes $V = B \cdot h$, which we have already proved to be correct (§§ 296, 314).

Outline of Proof for Pyramids and Cones. In these cases we know that the area of any section parallel to the base B is proportional to the square of its distance from the vertex (§§ 301, 322). Hence, since M is at a distance $h/2$ from the vertex,

$$\frac{M}{B} = \frac{(h/2)^2}{h^2} = \frac{1}{4}, \text{ or } B = 4\,M.$$

The top section T is zero, since the top bounding plane meets a pyramid or a cone in just one point on the vertex.

Hence the formula to be proved becomes, in this case,

$$V = [B + T + 4\,M] \cdot h/6 = (B + 0 + B) \cdot h/6 = Bh/3,$$

which we know to be correct. §§ 306, 323

Outline of Proof for Frustums. Given a frustum of a pyramid or of a cone, let H be the distance from the vertex O to the larger of the two bounding sections. Let B represent this larger section. Then $H - h$ is the distance from O to the other bounding section T, and $H - h/2$ is the distance from O to the middle section M.

We know that the volume V of the frustum is

$$V = [BH - T(H - h)]/3. \qquad \text{§§ 307, 324}$$

Or, since $T/B = (H - h)^2/H^2$, §§ 301, 322

we know that

$$V = B[H - (H - h)^3/H^2]/3 = \frac{B}{H^2}[3\,H^2 - 3\,Hh + h^2]h/3.$$

Since $T/B = (H - h)^2/H^2$, and $M/B = (H - h/2)^2/H^2$, the formula *to be proved* may be written,

$$V = [B + T + 4\,M]h/6 = \left[B + \frac{(H-h)^2}{H^2}B + 4\,\frac{(H-h/2)^2}{H^2}\,B\right]\frac{h}{6}$$

$$= \frac{B}{H^2}[3\,H^2 - 3\,Hh + h^2]\cdot\frac{h}{3}.$$

This is equivalent to the formula that we know to be correct; hence the theorem is proved.

337. Uses of the Prismoid Formula. The prismoid formula is a convenient means of remembering the volumes of a variety of solids. We shall see in Chapter VIII that it holds for spheres and frustums of spheres as well as for the solids of this chapter.

It also holds for any solid bounded by two parallel planes, made up by joining together pyramids, prisms, etc., bounded by the same two planes; such a solid is called a **prismoid**.

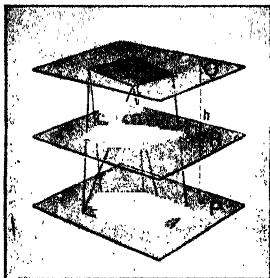


FIG. 229

The formula is used very extensively by engineers to estimate the volumes of various objects, such as the volume of a hill, or the volume of a metal casting. Since the same formula holds for such a large variety of solids, it is reasonably safe to use it *without even stopping to see which of these solids really resembles the object whose volume is desired.*

It is shown in more advanced books that *the same formula holds whenever the area of the section by any plane parallel to the bounding planes is proportional to the square of the distance from some fixed point to that plane.* Many solids not mentioned otherwise in elementary geometry satisfy this requirement.

MISCELLANEOUS EXERCISES. CHAPTER VII

1. Show that every section of a cylinder made by a plane passing through an element is a parallelogram. What is the section when the cylinder is a right cylinder?

2. Show that every section of a cone made by a plane through the vertex is a triangle. What is the section when the cone is a right cone?

3. If a, b, and c are the dimensions of a parallelepiped, show that the length of its diagonal is $\sqrt{a^2 + b^2 + c^2}$.

4. How long an umbrella will go into a trunk measuring 32 in. by 17 in. by 21 in., inside measure, (*a*) if the umbrella is laid on the bottom? (*b*) if it is placed diagonally between opposite corners of the top and bottom?

5. Find the volume of a pyramid whose base is a rhombus 6 in. on a side and whose height is 6 in., if one angle of the rhombus is 60°.

6. The Great Pyramid in Egypt is about 480 ft. high and its base is a square measuring about 764 ft. on a side. Find approximately its volume in cubic yards.

7. Water is poured into a cylindrical reservoir 25 ft. in diameter at the rate of 300 gallons a minute. Find the rate (number of inches per minute) at which the water rises in the reservoir (1 gal. = 231 cu. in.).

8. A copper teapot is $9\frac{7}{8}$ in. in diameter at the bottom, 8 in. at the top, and 11 in. deep. Allowing 42 sq. in. for locks and waste, how much metal is required for its construction, excluding the cover?

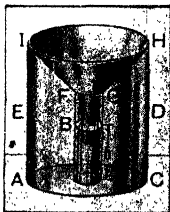
9. A conical spire has a slant height of 60 ft. and the perimeter of the base is 50 ft. Find the lateral surface.

10. How many cubic inches of lead are there in a piece of lead pipe 2 yd. long, the outer diameter being 2 in. and the thickness of the lead being $\frac{1}{4}$ of an inch?

11. The chimney of a factory has approximately the shape of a frustum of a regular pyramid. Its height is 75 ft. and its upper and lower bases are squares whose sides are 5 ft. and 8 ft. respectively. The flue is throughout a square whose side is 3 ft. How many cubic feet of material does the chimney contain? Assuming that a brick is 8 in. long, $3\frac{3}{4}$ in. wide, and $2\frac{1}{4}$ in. thick (as is ordinarily the case), estimate the number of bricks in such a chimney.

12. Compare the lateral areas, the total areas, and the volumes of (1) a right circular cylinder and a right circular cone having equal bases and altitudes, (2) a regular pyramid and a regular prism having congruent bases and equal altitudes.

13. A standard rain-gauge is made by inclosing a tube B in the interior of a can $ACDE$ and connecting the mouth of the tube to the mouth of the can by a funnel $FGHI$. The amount of water, measured in inches (depth), that has fallen in the vicinity of the gauge is determined by reading the height of the water in the tube B. Find a formula for the amount of rain that has fallen in terms of the height h of the water in the tube B, the radius r of the tube, and the radius R of the can. *Ans.* hr^2/R^2.



14. If one of the edges of a tetrahedron is 1 in. long, how long will be the corresponding edge of a similar tetrahedron of 8 times the volume? Answer the same question for the case in which the new tetrahedron is to have *half* the volume of the original. *Ans.* 2 in.; $1/\sqrt[3]{2} = .79$ in.

15. It is usual to state the diameter d of a tube in inches, and the area A of its surface in square feet. Show that the formula used by engineers:

$$A = 0.2618\, dl$$

gives very nearly the correct value in square feet, if d is measured in inches, and the length l is measured in feet.

CHAPTER VIII

THE SPHERE

PART I. GENERAL PROPERTIES

338. Spheres. A **sphere** is a portion of space bounded by a surface such that all straight lines to it from a fixed point within are equal.

The fixed point within the sphere is called its **center**; a line segment joining the center to any point on the surface is

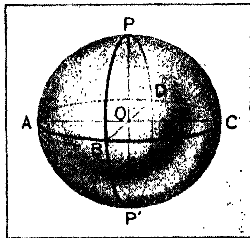


FIG. 230

a **radius**; a line segment drawn through the center and terminated at both ends by the surface is a **diameter**.

It follows from these definitions that:

(*a*) *The radii of a sphere, or equal spheres, are equal.*

(*b*) *The diameters of a sphere, or of equal spheres, are equal.*

(*c*) *Spheres having equal radii, or equal diameters, are equal.*

(*d*) *A sphere may be generated by the revolution of a semicircle about its diameter.*

EXERCISES

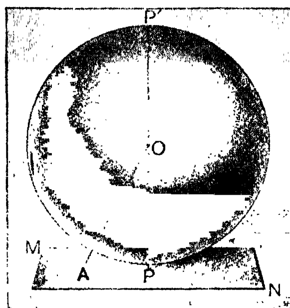
1. What is the locus of the points that are 2 in. from the surface of a sphere whose radius is 4 in.?

2. Show that the distance from the center of a sphere to a point outside the sphere is greater than the radius. (Use Ax. 10.) State the converse. Is it true?

3. If two spheres have the same center, they are called **concentric.** Show that one of two concentric spheres lies wholly within the other.

4. Show that if the center of each one of two given spheres lies on the surface of the other, their radii are equal.

5. Show by § 77, that a plane perpendicular to a diameter of a sphere at its extremity has only one point in common with the sphere.



339. Tangent Planes and Lines. A plane that has only one point in common with a sphere is called a **tangent plane** to the sphere. A line that has only one point in common with a sphere is called a **tangent line** to the sphere. In either case, the single common point is called the **point of tangency.**

340. Theorem I. *A plane perpendicular to a diameter of a sphere at one of its extremities is tangent to the sphere.*

Outline of Proof. Let MN be a plane perpendicular to a diameter PP' at P. Connect any point A of MN to the center of the sphere O. Show, by § 77, that $OA > OP$; whence, by § 338, A cannot be on the sphere, so that P is the *only* point of the plane on the sphere.

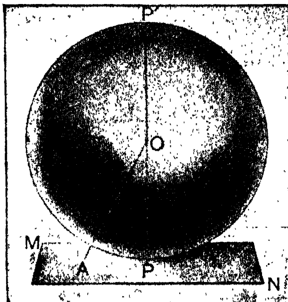


Fig. 231

341. Corollary 1. (*Converse of Theorem I.*) *If a plane is tangent to a sphere, it is perpendicular to the radius drawn to the point of contact.*

[Hint. Show, by Ax. 10, that the radius is shorter than any other line drawn from the center to the plane. Then use § 77.]

342. Corollary 2. *A straight line perpendicular to a diameter of a sphere at one of its extremities is tangent to the sphere; and conversely.* [Hint. Use §§ 254, 340 for direct, and § 116 for converse.]

343. Corollary 3. *All of the straight lines tangent to a sphere at a given point lie in the plane tangent to the sphere at that point.* [Hint. Use § 254.]

EXERCISES

1. What is the locus of a point in space at a given distance from a given point?

2. Prove that if two lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

[Hint. Connect this with one of the corollaries on this page.]

3. All lines tangent to a sphere from the same point are equal.

[Hint. Connect the center of the sphere with the given point and with two or more points of tangency.]

344. Theorem II. *Every section of a sphere made by a plane is a circle.*

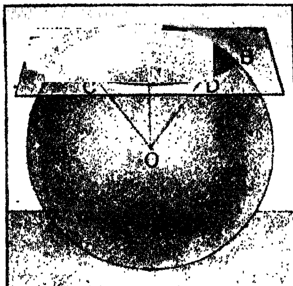


Fig. 232

Given the sphere whose center is O, cut by a plane in the section $AMBN$.

To prove that section $AMBN$ is a circle.

Proof. Draw $OQ \perp$ section $AMBN$; join Q to C and D, any two points in the perimeter of the section; draw OC and OD.

In the rt. ⧍ OQC and OQD,

$OQ = OQ$, and $OC = OD$. Why?

Therefore $\triangle OQC \cong \triangle OQD$. Why?

Therefore $QC = QD$. Why?

Since C and D are *any* two points on the perimeter of the section, all points on the perimeter of the section are equally distant from Q. Therefore section $AMBC$ is a circle. § 103

EXERCISES

1. If, in Fig. 232, the radius of the sphere, OC, is 10 in., and the distance OQ from the center O to the plane AB is 6 in., find the radius CQ of the circle.

2. If, in Fig. 232, the distances CQ and OC are given, show how to find the distance OQ.

345. Great and Small Circles. A circle on the sphere whose plane passes through the center is called a **great** circle of the sphere, as CD, Fig. 233.

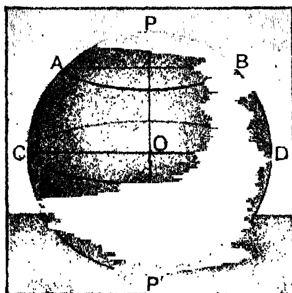


FIG. 233

A circle on the sphere whose plane does not pass through the center is called a **small circle** of the sphere, as AB, Fig. 233.

The **axis** of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle.

The **poles** of a circle of a sphere are the extremities of the axis of the circle.

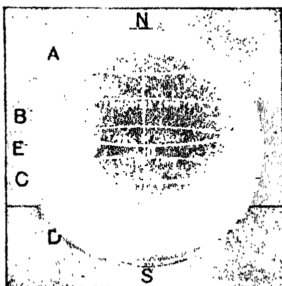
346. Corollary 1. *Through any three points on the surface of a sphere one and only one circle of the sphere may be drawn.*

[HINT. Use 4, § 241.]

347. Corollary 2. *Through any **two** points on the surface of a sphere a **great** circle may be drawn.* It follows from 4, § 241, that there is *one and only one* such great circle through the *two* given points, unless they lie at the opposite ends of a diameter.

348. Distance on a Sphere. By the **distance between two points** on the surface of a sphere is meant the length of the shorter arc of the great circle joining them. It can be shown that this is the shortest path on the surface of the sphere between the two points.

EXERCISES



1. If we consider the earth as a sphere, what kind of circles are the parallels of latitude? the equator? the meridians?

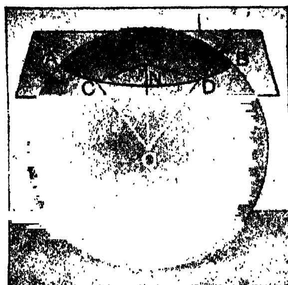
2. Prove that the axis of a small circle of a sphere passes through the center of the circle; and conversely, a diameter of the sphere through the center of a small circle is the axis of that small circle.

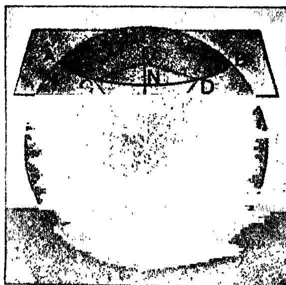
3. Prove that in the same sphere, or in equal spheres, all great circles are equal.

4. The radius of a sphere is 10 in. Find the area of a section made by a plane 5 in. from the center.

5. The area of a section of a sphere 7 in. from the center is 288 π sq. in. Find the area of a section 4 in. from the center.

6. Prove that in the same sphere, or in equal spheres, if two sections are equal, they are equally distant from the center, and conversely.





7. Prove that any two great circles of a sphere bisect each other.

349. Theorem III. *All points in the circumference of a circle of a sphere are equally distant from either one of its poles.*

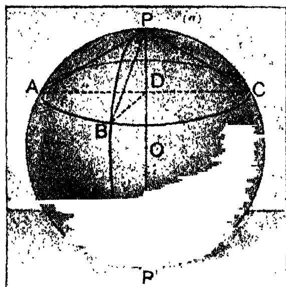


Fig. 234 (*a*)

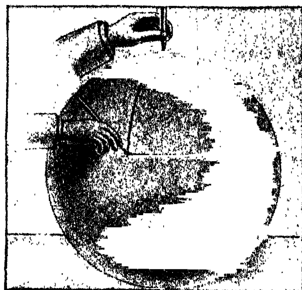


Fig. 234 (*b*)

Given any two points A and B in the circumference of the circle ABC, and P and P', the poles of ABC.

To prove that $\widehat{PA} = \widehat{PB}$, and $\widehat{P'A} = \widehat{P'B}$.

Proof. Draw the great circles PAP' and PBP'.

Let D be the intersection of the axis PP' with the plane ABC. Draw the straight lines AD, BD, PA and PB.

Now $PD = PD$, and $DA = DB$, Why?

and $\angle PDA = \angle PDB = 90°$. Why?

Hence chord $PA =$ chord PB, Why?

Therefore $\widehat{PA} = \widehat{PB}$. Why?

In the same way it may be proved that $\widehat{P'A} = \widehat{P'B}$.

Note. The manner in which circles may be drawn on a sphere is illustrated by Fig. 234 (*b*). If one end of a string is held at any point on the sphere, while a pencil attached to the other end is moved around the sphere, keeping the string taut, the end of the pencil describes a circle on the sphere, by Theorem III.

The various figures drawn in this chapter can be reproduced on the surface of an actual sphere, by this method of drawing the circles.

350. Polar Distance. The **polar distance** of a circle of a sphere is the distance on the sphere (§ 348) from its nearest pole to any point of the circumference, as $\widehat{PA}$ or $\widehat{PB}$ in Fig. 234.

A **quadrant** is one fourth part of the circumference of a great circle; *i.e.* an arc of 90° on a great circle.

351. Corollary 1. *The polar distance of a great circle is a quadrant.*

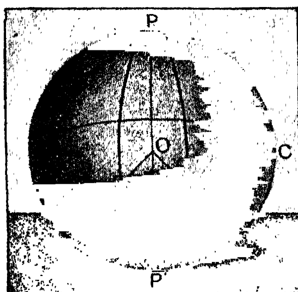


Fig. 235

[Hint. Let ABC be a great circle. Then its center O is also the center of the great circle PBP'. Hence the arc PB measures the right angle POB.]

EXERCISES

1. What is the locus of all the points on the surface of the earth at a quadrant's distance from the north pole? from the south pole? from the equator? See the figure for Ex. 1, p. 289.

2. The distance of the plane of a certain small circle from the center of a sphere is one half the radius of the sphere. If the diameter of the sphere is 12 in., find the polar distance of the small circle in degrees and in inches. *Ans.* 60°; 2π in.

3. Show that a great circle on the earth whose poles lie on the equator passes through the north pole.

352. Problem I. *To determine the radius of a given material sphere.*

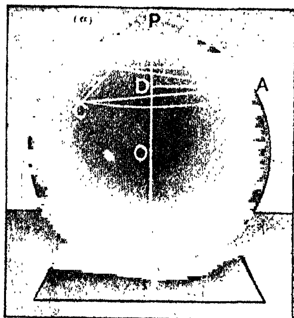


FIG. 236 (*a*)

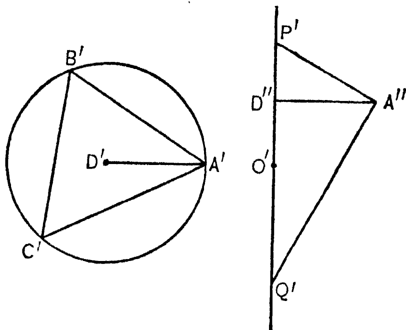


FIG. 236 (*b*)

Given any material sphere, *OPQ*.

To find its radius.

Construction. Take any point *P* on the surface of the sphere as a pole, and describe the circle *ABC*.

Take any three points on this circle, as *A*, *B*, *C*.

By means of the compasses construct on paper or on the blackboard the triangle $A'B'C'$ congruent to the triangle *ABC*.

Circumscribe a circle around $\triangle A'B'C'$, and let D' be the center of this circle.

Draw $D''A''$ equal to the radius $D'A'$.

Through D'' draw an indefinite line $P'Q'$ perpendicular to $D'A''$.

From A'' lay off with compasses $A''P'$ equal to line *AP*.

At A'' erect a perpendicular to $A''P'$ and extend it to meet $P'Q'$ at Q'.

Then $P'Q'$ is the diameter and $P'Q'/2$ is the radius of the given sphere.

[The proof is left to the student.]

353. Problem II. *To construct a sphere through four given points not all in the same plane.*

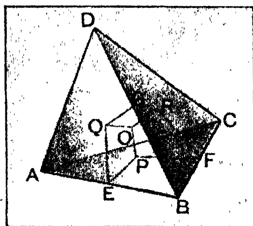


FIG. 237

Given the four points A, B, C, D not all in the same plane.

To construct a sphere that passes through A, B, C, and D.

Construction. At E, the middle point of AB, erect a plane QEP perpendicular to AB. Likewise, let PFR be a plane perpendicular to BC at its middle point F; and let QGR be a plane perpendicular to BD at its middle point G.

Let O be the point common to all three planes QEP, PFR, and QGR.

With O as center, and OA as radius, draw a sphere.

This is the required sphere passing through A, B, C, and D.

Proof. The plane QEP is the locus of all points equidistant from A and B. Why?

Likewise, PFR is the locus of points equidistant from B and C; and QGR is the locus of points equidistant from B and D.

The planes QEP and PFR meet in a line OP. Why?

The line OP meets the plane QGR in a single point O. Why?

Therefore O is equidistant from A, B, C, D. Why?

Moreover, O is the only point equidistant from A, B, C, D. Why?

354. Inscribed and Circumscribed Spheres. A sphere is said to be **circumscribed** about any polyhedron when the vertices of the polyhedron all lie on its surface.

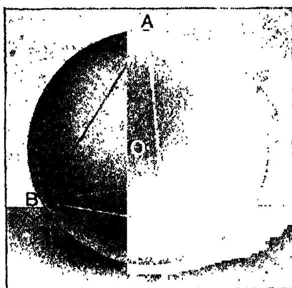


FIG. 238. CIRCUMSCRIBED SPHERE

A sphere is said to be **inscribed** in any polyhedron when it is tangent to each of the faces of the polyhedron.

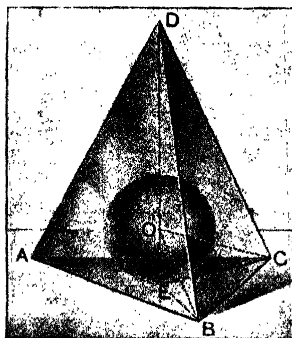


FIG. 239. INSCRIBED SPHERE

355. Corollary 1. *One and only one sphere may be circumscribed about any given tetrahedron (triangular pyramid).*

[HINT. Pass a sphere through the four vertices as in § 353. Notice that the four vertices cannot all lie in one plane.]

356. Problem III. *To inscribe a sphere in a given tetrahedron (triangular pyramid).*

Given the tetrahedron *ABCD*, Fig. 239.

To construct the sphere inscribed in it.

Construction and Proof. Bisect the dihedral angles whose edges are *BC*, *CD*, and *DB*, by the planes *BOC*, *CED*, and *DEB*, respectively.

The plane *BOC* is the locus of the points equidistant from the faces *BCD* and *BAC*; the plane *CED* is the locus of the points equidistant from the faces *BCD* and *CAD*; and the plane *DEB* is the locus of the points equidistant from the faces *BCD* and *DAB*. Why?

The intersection *O* of these planes is equidistant from the four faces of the tetrahedron. Hence the sphere whose center is *O* and whose radius is the perpendicular distance *OE* from *O* to the face *ABC*, is tangent to each of the faces; it is therefore inscribed in the tetrahedron.

No other sphere exists that is inscribed in the tetrahedron, for no other point than *O* is equidistant from the four faces. Why?

EXERCISES

1. By means of an instrument called a spherometer, the distances *AD* and *DP*, Fig. 236, can be measured directly. Show, by § 162, how to find the radius from these values.

2. Show that the process of § 352 can be used to find the radius of a sphere, if only a *piece* of the sphere is available, as in the case of a glass lens.

3. Show that four points in space determine a sphere, provided they do not lie in one plane.

4. Show that a sphere is determined if any circle that lies on it and one pole of that circle are given.

5. Show that any two circles of a sphere completely determine the sphere.

PART II. SPHERICAL ANGLES — TRIANGLES — POLYGONS

357. Spherical Angles. The line tangent to a great circle of a sphere at any point is a tangent to the sphere at that point; for it touches the sphere in only one point (§ 339).

The angle formed by the intersection of two great circles is called a **spherical angle.** It is defined to be equal to the angle formed by the tangents to the two great circles, at their point of intersection, as the angle *CPD*, Fig. 240.

358. Theorem IV. *The angle between two great circles is measured by the arc of a great circle described from its vertex as a pole and included between its sides.*

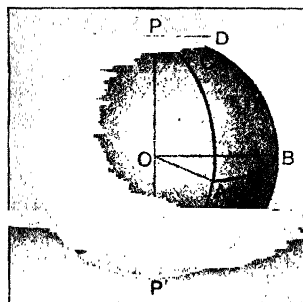


FIG. 240

Given the great circles PAP' and PBP' intersecting at P, and AB the arc of a great circle described with P as a pole.

To prove that $\widehat{AB}$ is the measure of $\angle APB$.

Proof. Draw the radii OA, OB, and the tangents PC, PD.

Then $OA \parallel PC$ and $OB \parallel PD$. Why?

Hence $\angle AOB = \angle CPD$. Why?

But $\angle AOB$ is measured by the arc AB; hence $\angle CPD$ is

measured by the arc AB. It follows that the spherical angle APB, which is equal to $\angle CPD$ by definition (§ 357), is measured by the arc AB.

359. Corollary 1. *The spherical angle between two great circles is equal to the plane angle of the dihedral angle formed by the planes of the two great circles.*

360. Spherical Triangles and Polygons. A **spherical polygon** is a portion of a spherical surface bounded by three or more arcs of **great circles**; as $ABCDE$, Fig. 241.

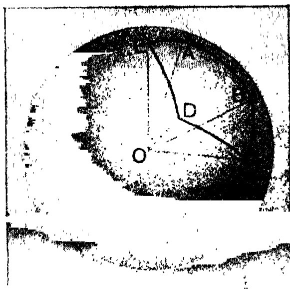


Fig. 241

The bounding arcs of great circles are called the **sides** of the spherical polygon; their intersections, the **vertices**; and the angles formed by the sides at the vertices, the **angles** of the spherical polygon.

A **diagonal** of a spherical polygon is an arc of a great circle joining any two non-adjacent vertices.

A **spherical triangle** is a spherical polygon of three sides, as ABC, Fig. 242.

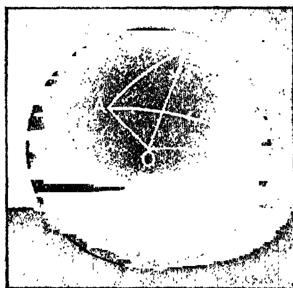


Fig. 242

The words **isosceles, equilateral, acute, right,** and **obtuse** are applied to spherical triangles in precisely the same way as to plane triangles.

Thus, in Fig. 242, the spherical triangle ABC is isosceles if the two sides, as AB and BC, are equal; the triangle is equilateral if $AB = BC = AC$; the triangle is a right triangle if any one angle is a right angle; etc.

361. Relation to Central Polyhedral Angles. The planes of the arcs of the great circles forming the sides of a spherical polygon meet at the center of the sphere and form a polyhedral angle, as *O–ABCD*.

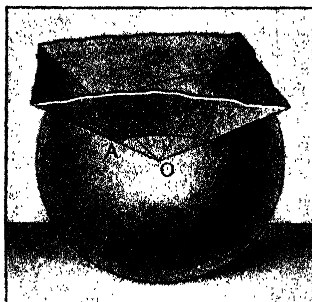


FIG. 243

This polyhedral angle and the spherical polygon are so closely related that the student can easily prove the following statements:

(*a*) *The sides of a spherical polygon have the same measures as the corresponding face angles of the polyhedral angle.*

(*b*) *The angles of the spherical polygon have the same measures as the corresponding dihedral angles of the polyhedral angle.*

Thus, sides *AB*, *BC*, etc., of the spherical polygon *ABCD* have the same measures as face ∠s *AOB*, *BOC*, etc., of polyhedral ∠ *O–ABCD*; and spherical ∠s *ABC*, *BCD*, etc., have the same measures as the dihedral ∠s whose edges are *OB*, *OC*, etc.

(*c*) *Any angle of a spherical polygon* (*or, the corresponding dihedral angle of the polyhedral angle*) *is measured by the arc of a great circle described with the vertex of the angle as pole and terminated by the sides.* See §§ 358, 359.

In general, *any fact proved for the sides and the angles of a spherical polygon is true also for the corresponding face angles and dihedral angles of the corresponding central polyhedral angle.*

362. Theorem V. *The sum of any two sides of a spherical triangle is greater than the third side.* [Compare § 272.]

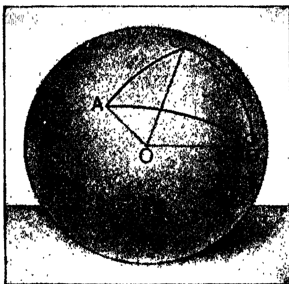


FIG. 244

Given the spherical △ ABC.

To prove that $\widehat{AB} + \widehat{BC} > \widehat{CA}$.

Proof. $\angle AOB + \angle BOC > \angle COA$. § 272

$\angle AOB$ is measured by $\widehat{AB}$,

$\angle BOC$ is measured by $\widehat{BC}$,

$\angle COA$ is measured by $\widehat{CA}$. Why?

Therefore $\widehat{AB} + \widehat{BC} > \widehat{CA}$.

363. Theorem VI. *The sum of the sides of any convex spherical polygon is less than 360°.* [Compare § 273.]

Given the spherical polygon $ABCD$.

To prove that

$\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA} < 360°$.

[HINT. Make use of § 273.]

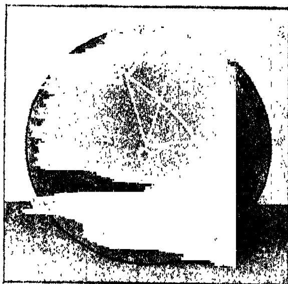


FIG. 245

EXERCISES

1. Show that any side of a spherical polygon is less than 180°.

2. In the spherical △ ABC, $\widehat{AB} = 35°$, and $\widehat{BC} = 75°$. Between what limits must $\widehat{CA}$ lie?

3. Three of the sides of a spherical quadrilateral are respectively 88° 17′, 70° 36′, and 50° 33′. Between what limits must the fourth side lie?

364. Polar Triangles. If from the vertices of a spherical triangle as poles arcs of great circles are drawn, these arcs form a second triangle which is called the **polar triangle** of the first.

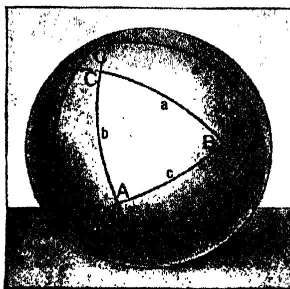


Fig. 246

Thus, if A, B, C, the vertices of the spherical △ ABC, are the poles of the arcs $B'C'$, $A'C'$, $A'B'$, forming the spherical △ $A'B'C'$, then $A'B'C'$ is the polar triangle of ABC.

If the entire circles be drawn, they will intersect so as to form eight spherical triangles, but the polar of the given triangle ABC is that one of the eight triangles whose vertices lie on the same side of the arcs of the given triangle as the corresponding vertices of the given triangle, and no side of which is greater than 180°.

365. Theorem VII. *If one spherical triangle is the polar of another, then the second is the polar of the first.*

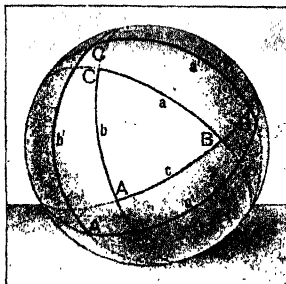


FIG. 247

Given $\triangle A'B'C'$, the polar of $\triangle ABC$.

To prove that $\triangle ABC$ is the polar of $\triangle A'B'C'$.

Proof. A is the pole of $\widehat{B'C'}$, and C is the pole of $\widehat{A'B'}$; Given.

hence B' is at a quadrant's distance from A and C, so that B' is the pole of $\widehat{AC}$. § 249

Similarly, A' is the pole of $\widehat{BC}$, and C' is the pole of $\widehat{AB}$.

Therefore ABC is the polar triangle of $A'B'C'$. § 364.

EXERCISES

1. Show that if one side of a spherical triangle on the earth's surface is on the equator, one vertex of the polar triangle is either at the north pole or at the south pole.

2. Show that if one vertex of a triangle on the earth is at the north pole, one side of the polar triangle is on the equator.

3. Show that if one vertex of a triangle on the earth is at the north pole, and if one side of the triangle is on the equator, the polar triangle also has one vertex at the north pole and one side along the equator.

366. Theorem VIII. *In two polar triangles, each angle of the one is measured by the supplement of the side opposite to it in the other.*

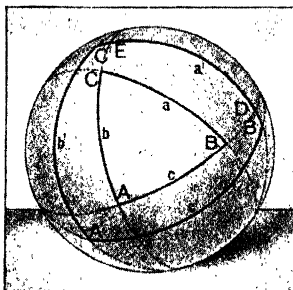


FIG. 248

Given the polar triangles ABC and $A'B'C'$, with the sides denoted by a, b, c, and a', b', c', respectively.

To prove that

(*a*) $\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$;
(*b*) $\angle A' + a = 180°$, $\angle B' + b = 180°$, $\angle C' + c = 180°$.

Proof. Let $\widehat{AB}$ and $\widehat{AC}$ (prolonged, if necessary) intersect $\widehat{B'C'}$ at D and E, respectively.

Then $\widehat{C'D} = 90°$, and $\widehat{EB'} = 90°$. Why?

Therefore $\widehat{C'D} + \widehat{EB'} = 180°$. Why?

That is $\widehat{C'E} + \widehat{ED} + \widehat{ED} + \widehat{DB'} = 180°$,

or, $\widehat{ED} + a' = 180°$.

But $\widehat{ED}$ is the measure of $\angle A$. (*c*) § 361

Therefore $\angle A + a' = 180°$.

In a similar manner $\angle B + b' = 180°$, and $\angle C + c' = 180°$.

The proof of (*b*) is left for the student.

EXERCISE

1. If the angles of a spherical triangle are 70°, 90°, and 80°, respectively, find the sides of the polar triangle (in degrees).

367. Theorem IX. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*

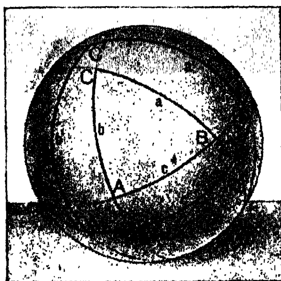


FIG. 249

Given the spherical $\triangle ABC$ with the sides a, b, and c.

To prove that $\angle A + \angle B + \angle C > 180°$ and $< 540°$.

Proof. Let $\triangle A'B'C'$, with its sides denoted by a', b', and c', be the polar of $\triangle ABC$.

Then $\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$.

Therefore $\angle A + \angle B + \angle C + a' + b' + c' = 540°$. Why?

But $a' + b' + c' < 360°$. § 363

Therefore $\angle A + \angle B + \angle C > 180°$. Why?

Again $a' + b' + c' > 0°$.

Therefore $\angle A + \angle B + \angle C < 540°$. Why?

368. Corollary 1. *In a spherical triangle there can be one, two, or even three right angles; there can be one, two, or three obtuse angles.*

EXERCISES

1. Show that a triangle on the earth's surface whose sides are the equator and two meridians, has two of its angles right angles, and two of its sides quadrants.

2. If, as in Ex. 1, two of the angles of a spherical triangle are right angles, between what limits must the third angle lie?

369. Birectangular and Trirectangular Triangles. A spherical triangle having two right angles is called a **birectangular** spherical triangle. A spherical triangle having all of its angles right angles is called a **trirectangular** spherical triangle.

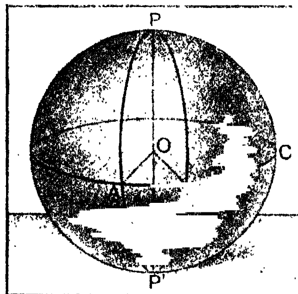


Fig. 250

If $\angle P$ in the figure is either acute or obtuse, while $\angle s\ A$ and B are right, $\triangle ABP$ is *birectangular;* if $\angle P$ is also a right angle, $\triangle ABP$ is *trirectangular,* as in Fig. 230, p. 284.

EXERCISES

1. The sides of a spherical triangle are 80°, and 126°, and 175°. How large are the angles of its polar triangle?

2. Show that in a birectangular triangle the sides opposite the right angles are quadrants.

3. Show that three mutually perpendicular planes through the center of a sphere divide its surface into eight congruent trirectangular triangles.

4. Show that the area of a trirectangular triangle on a sphere is one eighth of the area of the sphere.

5. Show that each of the sides of a trirectangular triangle is a quadrant. Hence show that the polar of a trirectangular triangle coincides with it.

370. Symmetric Triangles. Two spherical triangles are **symmetric** when their parts are equal each to each, but are in opposite order. Thus, in the $\triangle$ ABC and $A'B'C'$ (Fig. 251),

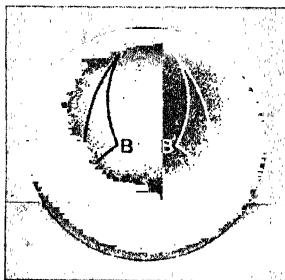


FIG. 251

if angles $A = A'$, $B = B'$, $C = C'$, and sides $AB = A'B'$, $BC = B'C'$, $CA = C'A'$, but the order of arrangement is opposite in the two figures, the triangles are symmetric.

In general, two symmetric triangles cannot be superposed and hence cannot be said to be congruent.

Thus, if $\triangle$ ABC is moved so that side AB coincides with its equal, $A'B'$, in the symmetric $\triangle$ $A'B'C'$, then the vertices C and C' lie on opposite sides of $A'B'$. In plane triangles, $\triangle$ ABC could be revolved about AB till it coincided with $\triangle$ $A'B'C'$; but this is in general impossible with spherical triangles.

371. Corollary. *Two* ***isosceles*** *symmetric spherical triangles are congruent.*

EXERCISES

1. Prove that the base angles of an isosceles spherical triangle are equal.

[HINT. Draw an arc bisecting the vertical angle, thus forming two symmetric triangles.]

2. Show that if two sides of a spherical triangle are quadrants, the triangle is birectangular.

372. Theorem X. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric, if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.*

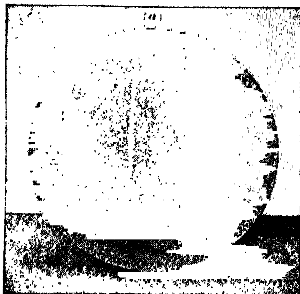


FIG. 252 (*a*)

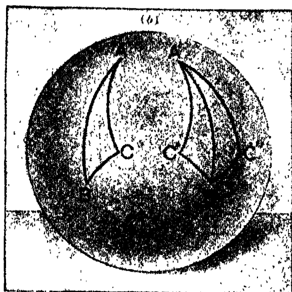


FIG. 252 (*b*)

Given the spherical $\triangle$ ABC and $A'B'C'$ on the same sphere or equal spheres, having $AB = A'B'$, $AC = A'C'$, $\angle A = \angle A'$.

To prove that $\triangle$ ABC and $A'B'C'$ are either congruent or else symmetric.

Proof. If the equal parts of the two triangles are in the same order, $\triangle ABC$ can be placed on $\triangle A'B'C'$ as in the corresponding case of plane triangles. See Fig. 252 (*a*).

If the equal parts of the two triangles are not in the same order, construct $\triangle A'B'C''$ symmetric to $\triangle A'B'C'$. (Fig. 252 (*b*).)

In $\triangle$ ABC and $A'B'C''$, $AC = A'C''$, $AB = A'B'$, and $\angle A = \angle B'A'C''$. Since these parts are arranged in the same order, $\triangle$ ABC and $A'B'C''$ are congruent. Therefore spherical $\triangle ABC$ is symmetric to spherical $\triangle A'B'C'$. Why?

373. Theorem XI. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric, if two angles and the included side of the one are equal, respectively, to two angles and the included side of the other.* [Proceed as in § 372.]

374. Theorem XII. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric, if the three sides of the one are equal, respectively, to the three sides of the other.*

[The proof is left to the student.]

375. Theorem XIII. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric, if the three angles of the one are equal, respectively, to the three angles of the other.*

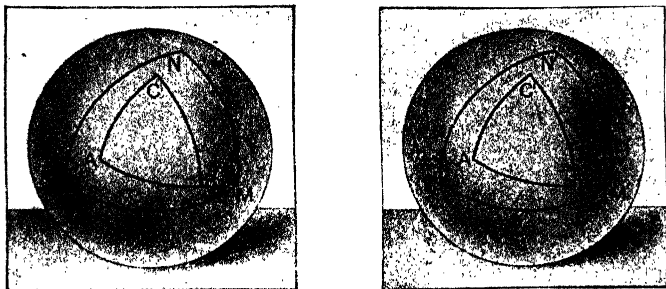


FIG. 253

Outline of Proof. If ABC and $A'B'C'$ are the two given spherical triangles so that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, their *polar* triangles LMN and $L'M'N'$ have the three *sides* of one equal to the three sides of the other, respectively. § 366

Then, by § 374, △ LMN and $L'M'N'$ are either congruent or symmetric. In either case, the three angles of $\triangle LMN$ are equal to the three angles of $\triangle L'M'N'$, respectively; and therefore the three sides of $\triangle ABC$ are equal to the three sides of $\triangle A'B'C'$, respectively. § 366

It follows, by § 374, that △ ABC and $A'B'C'$ are either congruent or symmetric.

NOTE. Theorems analogous to those of §§ 41, 43, 44, etc., may be proved in a manner similar to §§ 372–375.

EXERCISES

1. Show that two trihedral angles are congruent if they intercept congruent triangles on the surfaces of two equal spheres whose centers are at their vertices, respectively.

2. If two trihedral angles intercept symmetric spherical triangles on the surface of a sphere whose center is at their vertices, respectively, show that the face angles and the dihedral angles of one trihedral angle are equal to those of the other, but taken in reversed order.

[Such trihedral angles are called **symmetric.**]

3. Prove the following theorem, which states for trihedral angles (§ 361) a theorem analogous to that of § 372:

Two trihedral angles are either congruent or symmetric if two face angles and the included dihedral angle of the one are respectively equal to two face angles and the included dihedral angle of the other.

[HINT: Consider the spherical triangles cut out by the two trihedral angles on the surfaces of two equal spheres whose centers lie at the vertices of the two trihedral angles, and apply § 372.]

4. Prove the following theorem, analogous to § 373:

Two trihedral angles are either congruent or symmetric if two dihedral angles and the included face angle of the one are respectively equal to two dihedral angles and the included face angle of the other.

5. State and prove theorems for trihedral angles similar to Theorems XII–XIII, § 374–375.

6. Prove Theorem XI, § 373, by first considering, as in § 375, the *polars* of the given triangles, and applying § 372.

7. Show that any trirectangular triangle on the earth's surface is congruent to the trirectangular triangle formed by the equator and two meridians whose longitude differs by 90°.

PART III. AREAS AND VOLUMES

376. Theorem XIV. *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis and the length of the circle whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.*

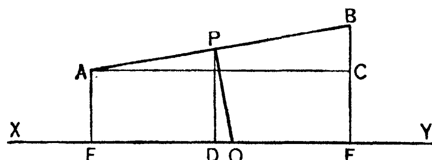


FIG. 254

Given EF, the projection upon XY of AB revolving about XY, and $OP \perp AB$ at its mid-point, and meeting XY at O.

To prove that the area generated by $AB = EF \times 2\,\pi OP$.

Proof. Draw $PD \perp XY$, and $AC \parallel XY$.

Since the surface generated by AB is the lateral surface of the frustum of a cone, the area generated by AB is

$$\frac{AB}{2}\left(2\,\pi AE + 2\,\pi BF\right) = AB \times 2\,\pi \cdot \left(\frac{AE+BF}{2}\right)$$

$$= AB \times 2\,\pi \cdot PD. \qquad \S\ 321$$

Now $\triangle ABC \sim \triangle POD.$ § 157

Therefore $AB : OP = AC : PD.$ Why?

Then $AB \times PD = AC \times OP = EF \times OP.$ Why?

And $AB \times 2\,\pi PD = EF \times 2\,\pi OP.$

That is, the area generated by AB is $EF \times 2\,\pi OP$.

If AB meets XY, the surface generated is a conical surface whose area again $= EF \times 2\,\pi OP$. § 320

If AB is parallel to XY, the surface generated is a cylindrical surface whose area again $= EF \times 2\,\pi OP$. § 313

377. Theorem XV. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

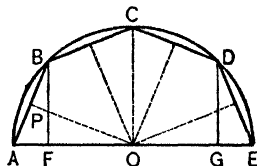


FIG. 255

Given a sphere generated by the revolution of the semicircle $ABCDE$ about the diameter AOE, S being the area of the surface, r being the radius, and d being the diameter.

To prove that $S = 2\,\pi rd.$

Proof. Inscribe in the semicircle half of a regular polygon $ABCDE$, of any number of sides, and draw BF, CO, DG, perpendicular to AE.

From O draw $OP \perp AB$. Then OP bisects AB, Why? and is equal to each of the ⊥s drawn from O to the equal chords BC, CD, DE. Why?

Now the area generated by $AB = AF \times 2\,\pi \cdot OP$, § 376
the area generated by $BC = FO \times 2\,\pi \cdot OP$,
the area generated by $CD = OG \times 2\,\pi \cdot OP$,
the area generated by $DE = GE \times 2\,\pi \cdot OP$.

Therefore, if S' denotes the surface generated by the semipolygon,

$$S' = (AF + FO + OG + GE)\,2\,\pi OP = AE \times 2\,\pi OP.$$

Let the number of sides of the semipolygon be now indefinitely increased.

Then OP has for its limit r, the semipolygon for its limit the semicircle, and S' for its limit S. Hence, as in § 303,

$$S = AE \times 2\,\pi r.$$

378. Corollary 1. The area of the surface of a sphere is equal to $4\,\pi r^2$.

379. Corollary 2. *The area of the surface of a sphere is equal to the sum of the areas of four great circles.*

For $S = 2r \times 2\pi r = 4\pi r^2$, § 378

and πr^2 is the area of a great circle.

380. Corollary 3. *The areas of the surfaces of two spheres are to each other as the squares of their radii; or, as the squares of their diameters.*

381. Zones. A **zone** is a portion of the surface of a sphere bounded by the circumferences of two circles whose planes are parallel.

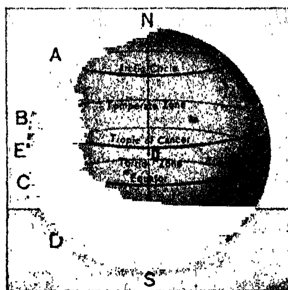


FIG. 256. ZONES ON THE EARTH'S SURFACE

The circumferences forming the boundary of a zone are its **bases.**

If the semicircle NES is revolved about NS as an axis, arc AB will generate a zone, while points A and B will generate the bases of the zone.

The **altitude** of a zone is the perpendicular distance between the planes of the bases.

382. Corollary 4. *The area of a zone of a sphere is equal to the product of the altitude h of the zone and the circumference of a great circle; or $2\pi rh$, where r is the radius of the sphere.*

383. Lunes. A **lune** is a portion of a spherical surface bounded by two semicircumferences of great circles; as

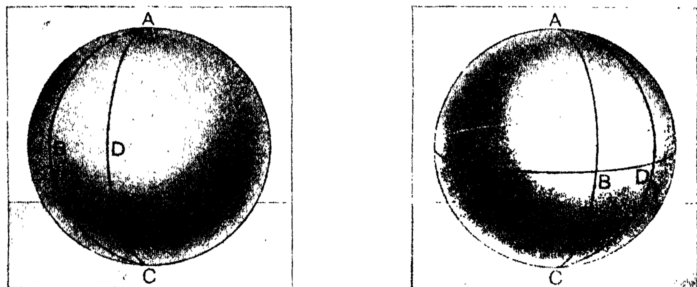


Fig. 257

ABCDA (Fig. 257). The **angle of a lune** is the angle formed by its bounding arcs. Thus *BAD* is the angle of the lune *ABCDA*.

384. Theorem XVI. *The area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles.*

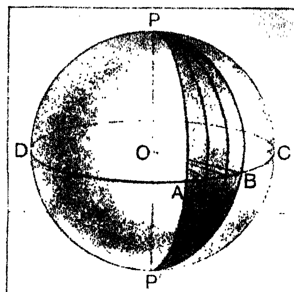


Fig. 258

Given the lune *PAP'B*, let *L* denote the area of the lune, *S* the area of the surface of the sphere, and *a* the angle of the lune.

To prove that $L/S = \angle a/4$ rt. $\angle$s.

Proof. With P as a pole describe the great circle $ABCD$.

Then the arc AB measures $\angle a$ of the lune. Why ?

Therefore arc AB/circle $ABCD = \angle a/4$ rt. $\angle$s.

If AB and $ABCD$ are commensurable, let their common measure be contained m times in AB and n times in $ABCD$.

Then arc AB/circle $ABCD = m/n$.

Therefore $a/4$ rt. $\angle$s $= m/n$. § 358

Pass arcs of great circles through each point of division of $ABCD$ and the poles P and P'.

These arcs will divide the entire surface into n equal lunes, of which $PAP'B$ will contain m.

Therefore $L/S = m/n$,

or, $L/S = a/4$ rt. $\angle$s.

If AB and $ABCD$ are incommensurable, the theorem can be proved as in § 130. The details are left to the student.

385. Corollary 1. *The area of a lune whose angle is* $1°$ *is* $4\,\pi r^2/360 = \pi r^2/90$.

386. Corollary 2. *The area of a lune whose angle is* $k°$ *is* $4\,\pi r^2 k/360 = \pi r^2 k/90$.

EXERCISES

1. If the surface of a sphere is 10 sq. ft., what is the area of a lune whose angle is 40° ? What is the radius of the sphere? *Ans.* $1\frac{1}{9}$ sq. ft.; 0.89^+ ft.

2. Show that two lunes on the same sphere or equal spheres have the same ratio as their angles.

3. What is the angle of a lune which has the same area as a trirectangular triangle ?

4. Show that the area of a lune is one ninetieth of the area of a great circle multiplied by the number of degrees in the angle of the lune.

387. Theorem XVII. *Two symmetric triangles are equal in area.*

Given the two symmetric spherical triangles ABC and $A'B'C'$.

To prove that

$$\triangle ABC = \triangle A'B'C'.$$

Proof. Let P be the pole of the small circle passing through the points A, B, C, and draw the great circle arcs PA, PB, and PC.



Fig. 259

Then $PA = PB = PC$. Why?

Now place the two triangles diametrically opposite to each other and draw the diameter POP'. Also draw the great circle arcs $P'A'$, $P'B'$, and $P'C'$. Then the triangles PBC and $P'B'C'$ are symmetrical and isosceles and therefore congruent. § 371.

Similarly $\triangle PCA \cong \triangle P'C'A'$,

and $\triangle PAB \cong \triangle P'A'B'$.

That is, the three parts of ABC are respectively congruent to the three parts of ABC.

Therefore $\triangle ABC = \triangle A'B'C'$.

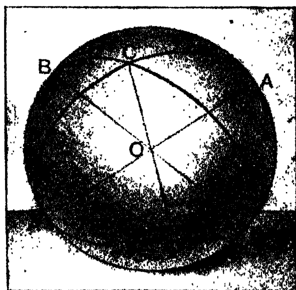


Fig. 260

388. Corollary 1. *If two semicircumferences of great circles BCB' and ACA' intersect on the surface of a hemisphere, the sum of the areas of the two opposite spherical triangles ACB and $A'CB'$ is equal to the area of a lune whose angle is equal to ACB.*

[Hint. Show that the triangle ABC is symmetric to the triangle $A'B'C'$. Hence show that $\triangle ACB + \triangle A'CB' =$ lune $A'CB'C'$.]

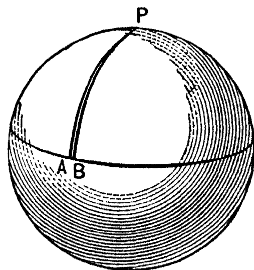
389. Spherical Degree. The area of a lune whose angle is 1° is $4\pi r^2/360$, or $\pi r^2/90$ (§ 385). Half this area, that is, $4\pi r^2/720$ or $\pi r^2/180$, is often *taken as a unit of area on the sphere,* and it is called a **spherical degree.**

390. Measure of Solid Angles. A trihedral angle whose vertex is at the center of a sphere cuts out a spherical triangle on the surface of the sphere. The area of the spherical triangle, in spherical degrees, is called the **measure** of the trihedral angle.

Likewise any polyhedral angle is measured by the area, in spherical degrees, that it cuts out upon the surface of a sphere whose center is at its vertex.

EXERCISES

1. Show that the area of a lune whose angle is 1° is 2 spherical degrees.



2. Show that the area of the entire sphere is 720 spherical degrees

3. Show that the area of a birectangular triangle whose third angle is 1° is 1 spherical degree.

4. Show that the area of a trirectangular triangle is 90 spherical degrees, or one eighth of the entire surface.

391. Spherical Excess. The excess of the sum of the angles of a spherical triangle over 180° is called the **spherical excess** of the triangle.

If, for example, the angles of a spherical triangle are 80°, 100°, and 125°, the spherical excess of the triangle is 125°.

Likewise, the spherical excess of any spherical polygon is the excess of the sum of its angles above the sum of the angles of a plane polygon of the same number of sides.

392. Theorem XVIII. *The area of a spherical triangle is equal to the area of a lune whose angle is half the spherical excess of the triangle.*

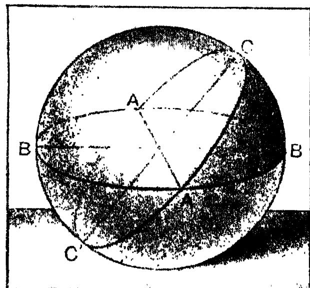


FIG. 261

Given the spherical $\triangle ABC$.

To prove that $\triangle ABC$ is equal to a lune whose angle is

$$\tfrac{1}{2}(\angle A + \angle B + \angle C - 180°).$$

Proof. Complete the great circles by producing the sides of the $\triangle ABC$, as in Fig. 261.

Since $\triangle$s $AB'C'$ and $A'BC$ are symmetric, they are equal in area § 387

Therefore lune $ABA'C = \triangle ABC + \triangle AB'C'$. § 388

But, denoting the area of the whole sphere by S,

$$\triangle CB'A + \triangle AC'B + \triangle ABC + \triangle AB'C' = \tfrac{1}{2} S. \quad \text{Why?}$$

Therefore

$$(\text{lune } BCB'A - \triangle ABC) + (\text{lune } CAC'B - \triangle ABC)$$
$$+ \text{lune } ABA'C = \tfrac{1}{2} S. \quad \text{Why?}$$

Therefore, transposing, we obtain

$$2 \triangle ABC = \text{lune } ABA'C + \text{lune } BCB'A + \text{lune } CAC'B - \tfrac{1}{2} S.$$

But $\frac{1}{2} S$ is the area of a lune whose angle is 180°.

Therefore $\triangle ABC$ is equal to a lune whose angle is

$$\tfrac{1}{2}(\angle A + \angle B + \angle C - 180°). \quad \text{§ 384}$$

393. Corollary 1. *The area of a spherical triangle, measured in spherical degrees, is numerically equal to its spherical excess.*

NOTE. This result enables us to compute the area of any spherical triangle in ordinary units of area, when we know its angles and the radius of the sphere. Thus, if r denotes the radius of the sphere, E the spherical excess, and A the required area, we have, by § 385

$$A = E \times \frac{\pi r^2}{180} = \frac{\pi r^2 E}{180}.$$

394. Corollary 2. *The area of a trirectangular triangle is 90 spherical degrees.*

395. Corollary 3. *The area of any spherical triangle is to the area of the entire sphere as its spherical excess is to 720°.*

396. Solid Angles. In general, if any closed polygon or curve is drawn on the surface of a sphere, the figure formed by all radii of the sphere that join the center to the points of this figure on the spherical surface is called a **solid angle.** The area on the surface of the sphere cut out by such a solid angle, in spherical degrees, is the **measure** of the solid angle.

EXERCISES

1. What is the measure of a hemisphere in spherical degrees?

2. The radius of a sphere is 2 ft. Find the area of a triangle on its surface whose angles are 75°, 35°, 105°, respectively. Solve first by § 392; then by § 394. *Ans.* $7\pi/9$ sq. ft.

3. The radius of the earth is approximately 4000 miles. Find the entire area. Show that the area in square miles of one spherical degree is approximately 278,000 square miles.

4. Find how large a triangle on the earth's surface would have the total sum of its three angles equal to 181°.

5. Show that a region containing about 270,000 sq. mi. on the earth contains no triangle whose spherical excess is 1°.

6. What is the area of the state in which you live? What is its measure in spherical degrees?

397. Theorem XIX. ***The volume V of a sphere is equal to the product of its surface by one third of its radius; or,*** $V = 4\,\pi r^3/3$.

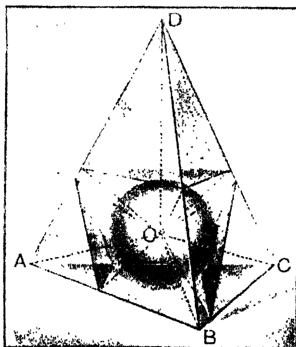


FIG. 262

Given a sphere whose center is O; let S denote its surface, r its radius, and V its volume.

To prove that $\quad V = S \times r/3 = 4\,\pi r^3/3$

Proof. Circumscribe about the sphere any polyhedron as $D–ABC$, and denote its surface by S' and its volume by V'.

Form pyramids, as $O–ABC$, etc., having the faces of the polyhedron as bases and the center of the sphere as a common vertex.

These pyramids will have a common altitude equal to r, and the volume of each pyramid is equal to its *base* $\times\ r/3$. Why?

Therefore $\quad V' = S' \times r/3.$ Why?

If the number of pyramids is indefinitely increased by passing planes tangent to the sphere at points where the edges of the pyramids cut the surface of the sphere, as in Fig. 262, the difference between S and S' becomes as small as we please; the difference between V and V' becomes as small as we please.

But however great the number of pyramids,

$$V' = S' \times r/3.$$

Therefore, as in § 303, $V = S \times r/3.$

Since $S = 4\,\pi r^2,$

it follows that $V = 4\,\pi r^2 \times r/3 = 4\,\pi r^3/3.$

398. Corollary 1. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

399. Corollary 2. *The volume of the pyramidal piece cut out of a sphere by any polyhedral angle whose vertex is at the center is equal to one third the area of the spherical polygon cut out of the surface times the radius.*

400. Corollary 3. *The prismoid formula* (§ 336) *holds for a sphere.*

[HINT. Two parallel planes that include the entire sphere are tangent planes at the ends of a diameter; these cut the sphere in only one point each. A plane parallel to these two and halfway between them cuts the sphere in a great circle. Hence, in the notation of § 336, $B = 0$, $T = 0$, $M = 4\,\pi r^2$, $h = 2\,r$; hence the prismoid formula would give

$$V = h\left[\frac{B + T + 4\,M}{6}\right] = 2\,r\left[\frac{0 + 0 + 4\,\pi r^2}{6}\right] = \frac{4\,\pi r^3}{3},$$

which, by § 397, is correct.]

[NOTE. As a matter of fact, the prismoid formula holds for the portion of a sphere intercepted between any two parallel planes.]

EXERCISES

1. Assuming that the earth is a sphere whose radius is 4000 mi., find its volume.

2. Show that a cube circumscribed about a sphere has a volume $8\,r^3$. Hence show that the sphere occupies a little more than half this volume

3. Find the volume of the material in a hollow sphere, if the radius of the outer surface is 6 in. and that of the inner surface is 5 in.

4. Show that the volume of a hollow sphere whose outer and inner radii are R and r, respectively, is $4\pi(R^3 - r^3)/3$.

5. Find the volume of the material in a hollow sphere whose outer radius is 10 in., if the material is $\frac{1}{2}$ in. thick.

6. Show that the volume of a sphere in terms of its diameter, d, is $\pi d^3/6$.

7. If the radius of one sphere is twice that of another, how do their volumes compare?

8. If the volume of one sphere is twice that of another, how do their radii compare?

9. Find approximately the radius of a sphere whose volume is 100 cu. in.

10. How many shot $\frac{1}{10}$ in. in diameter can be made from 10 cu. in. of lead?

11. If oranges 3 in. in diameter sell for 30 cents per dozen, and those 4 in. in diameter sell for 50 cents per dozen, which are the cheaper by volume?

12. If the skins are of equal thickness, which of the oranges of Ex. 11 has the greater percentage of skin to the cubic inch of volume?

13. Assuming that raindrops are practically spherical, if the diameter of one drop is half that of another, how do their volumes compare? their areas?

14. Which of the two drops of Ex. 13 has the greater ratio of area to volume? How much greater? Which will fall the more rapidly through the air?

[HINT. The greater the ratio of area to volume, for the same material, the slower the body will fall through the air.]

15. Explain, by the principle of Ex. 14, why very small dust particles remain floating in the air for a long time.

16. The same amount of material, in the form of a cube, is melted and cast into a sphere. Is the surface area less in the form of the cube or in that of the sphere?

[HINT. Assume the cube to be 1 unit on each edge; find the radius of the resulting sphere.]

17. If a surveyor wishes to be certain that the sum of the angles in any triangle in a region on the earth's surface shall be equal to 180° to within one minute, how large may the region be? *Ans.* About 4600 sq. mi.

18. Demonstrate the existence of spherical triangles with three obtuse angles from the existence of triangles whose sides are very short.



TABLES

TABLE I

Ratios of the Sides of Right Triangles
and
Chords and Arcs of a Unit Circle

TABLE II

Squares and Square Roots of Numbers
Cubes and Cube Roots of Numbers

TABLE III

Values of Important Numbers
including
Units of Measurement

TABLE I

RATIOS OF THE SIDES OF RIGHT TRIANGLES

AND

LENGTHS OF CHORDS AND ARCS OF A UNIT CIRCLE

EXPLANATION OF TABLE I

1. Ratios of the Sides of Right Triangles. If an angle given in the ***Angle Column*** is one acute angle of a ***right triangle:***

The Sine Column gives the ratio of the side opposite the angle to the hypotenuse;

The Tangent Column gives the ratio of the side opposite the angle to the side adjacent to the angle.

To find the ***Cosine*** of any angle, take the *sine of the complement* of that angle.

2. Chords and Arcs of a Unit Circle. If an angle given in the ***Angle Column*** is an angle at the center of a ***circle of unit radius:***

The Chord Column gives the length of the chord that subtends that angle;

The Arc Column gives the length of the arc that subtends that angle.

To find the ***lengths of chords or arcs of any circle of radius* r**, multiply the values given in the table by that radius.

The table is limited to angles less than 90°; but to find ***the chord that subtends an obtuse angle,*** first take half the angle, find the sine of this half angle, and multiply by 2. This follows from the fact that the chord of any angle is *twice the sine of half that angle.*

Angle	Sine	Tan-gent	Chord	Arc	Angle	Sine	Tan-gent	Chord	Arc
0° 00′	.0000	.0000	.0000	.0000	**9° 00′**	.1564	.1584	.1569	.1571
10	.0029	.0029	.0029	.0029	10	.1593	.1614	.1598	.1600
20	.0058	.0058	.0058	.0058	20	.1622	.1644	.1627	.1629
30	.0087	.0087	.0087	.0087	30	.1650	.1673	.1656	.1658
40	.0116	.0116	.0116	.0116	40	.1679	.1703	.1685	.1687
50	.0145	.0145	.0145	.0145	50	.1708	.1733	.1714	.1716
1° 00′	.0175	.0175	.0175	.0175	**10° 00′**	.1736	.1763	.1743	.1745
10	.0204	.0204	.0204	.0204	10	.1765	.1793	.1772	.1774
20	.0233	.0233	.0233	.0233	20	.1794	.1823	.1801	.1804
30	.0262	.0262	.0262	.0262	30	.1822	.1853	.1830	.1833
40	.0291	.0291	.0291	.0291	40	.1851	.1883	.1859	.1862
50	.0320	.0320	.0320	.0320	50	.1880	.1914	.1888	.1891
2° 00′	.0349	.0349	.0349	.0349	**11° 00′**	.1908	.1944	.1917	.1920
10	.0378	.0378	.0378	.0378	10	.1937	.1974	.1946	.1949
20	.0407	.0407	.0407	.0407	20	.1965	.2004	.1975	.1978
30	.0436	.0437	.0436	.0436	30	.1994	.2035	.2004	.2007
40	.0465	.0466	.0465	.0465	40	.2022	.2065	.2033	.2036
50	.0494	.0495	.0494	.0495	50	.2051	.2095	.2062	.2065
3° 00′	.0523	.0524	.0524	.0524	**12° 00′**	.2079	.2126	.2091	.2094
10	.0552	.0553	.0553	.0553	10	.2108	.2156	.2119	.2123
20	.0581	.0582	.0582	.0582	20	.2136	.2186	.2148	.2153
30	.0610	.0612	.0611	.0611	30	.2164	.2217	.2177	.2182
40	.0640	.0641	.0640	.0640	40	.2193	.2247	.2206	.2211
50	.0669	.0670	.0669	.0669	50	.2221	.2278	.2235	.2240
4° 00′	.0698	.0699	.0698	.0698	**13° 00′**	.2250	.2309	.2264	.2269
10	.0727	.0729	.0727	.0727	10	.2278	.2339	.2293	.2298
20	.0756	.0758	.0756	.0756	20	.2306	.2370	.2322	.2327
30	.0785	.0787	.0785	.0785	30	.2334	.2401	.2351	.2356
40	.0814	.0816	.0814	.0814	40	.2363	.2432	.2380	.2385
50	.0843	.0846	.0843	.0844	50	.2391	.2462	.2409	.2414
5° 00′	.0872	.0875	.0872	.0873	**14° 00′**	.2419	.2493	.2437	.2443
10	.0901	.0904	.0901	.0902	10	.2447	.2524	.2466	.2473
20	.0929	.0934	.0931	.0931	20	.2476	.2555	.2495	.2502
30	.0958	.0963	.0960	.0960	30	.2504	.2586	.2524	.2531
40	.0987	.0992	.0989	.0989	40	.2532	.2617	.2553	.2560
50	.1016	.1022	.1018	.1018	50	.2560	.2648	.2582	.2589
6° 00′	.1045	.1051	.1047	.1047	**15° 00′**	.2588	.2679	.2611	.2618
10	.1074	.1080	.1076	.1076	10	.2616	.2711	.2639	.2647
20	.1103	.1110	.1105	.1105	20	.2644	.2742	.2668	.2676
30	.1132	.1139	.1134	.1134	30	.2672	.2773	.2697	.2705
40	.1161	.1169	.1163	.1164	40	.2700	.2805	.2726	.2734
50	.1190	.1198	.1192	.1193	50	.2728	.2836	.2755	.2763
7° 00′	.1219	.1228	.1221	.1222	**16° 00′**	.2756	.2867	.2783	.2793
10	.1248	.1257	.1250	.1251	10	.2784	.2899	.2812	.2822
20	.1276	.1287	.1279	.1280	20	.2812	.2931	.2841	.2851
30	.1305	.1317	.1308	.1309	30	.2840	.2962	.2870	.2880
40	.1334	.1346	.1337	.1338	40	.2868	.2994	.2899	.2909
50	.1363	.1376	.1366	.1367	50	.2896	.3026	.2927	.2938
8° 00′	.1392	.1405	.1395	.1396	**17° 00′**	.2924	.3057	.2956	.2967
10	.1421	.1435	.1424	.1425	10	.2952	.3089	.2985	.2996
20	.1449	.1465	.1453	.1454	20	.2979	.3121	.3014	.3025
30	.1478	.1495	.1482	.1484	30	.3007	.3153	.3042	.3054
40	.1507	.1524	.1511	.1513	40	.3035	.3185	.3071	.3083
50	.1536	.1554	.1540	.1542	50	.3062	.3217	.3100	.3113
9° 00′	.1564	.1584	.1569	.1571	**18° 00′**	.3090	.3249	.3129	.3142

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
18° 00′	.3090	.3249	.3129	.3142	**27° 00′**	.4540	.5095	.4669	.4712
10	.3118	.3281	.3157	.3171	10	.4566	.5132	.4697	.4741
20	.3145	.3314	.3186	.3200	20	.4592	.5169	.4725	.4771
30	.3173	.3346	.3215	.3229	30	.4617	.5206	.4754	.4800
40	.3201	.3378	.3244	.3258	40	.4643	.5243	.4782	.4829
50	.3228	.3411	.3272	.3287	50	.4669	.5280	.4810	.4858
19° 00′	.3256	.3443	.3301	.3316	**28° 00′**	.4695	.5317	.4838	.4887
10	.3283	.3476	.3330	.3345	10	.4720	.5354	.4867	.4916
20	.3311	.3508	.3358	.3374	20	.4746	.5392	.4895	.4945
30	.3338	.3541	.3387	.3403	30	.4772	.5430	.4923	.4974
40	.3365	.3574	.3416	.3432	40	.4797	.5467	.4951	.5003
50	.3393	.3607	.3444	.3462	50	.4823	.5505	.4979	.5032
20° 00′	.3420	.3640	.3473	.3491	**29° 00′**	.4848	.5543	.5008	.5061
10	.3448	.3673	.3502	.3520	10	.4874	.5581	.5036	.5091
20	.3475	.3706	.3530	.3549	20	.4899	.5619	.5064	.5120
30	.3502	.3739	.3559	.3578	30	.4924	.5658	.5092	.5149
40	.3529	.3772	.3587	.3607	40	.4950	.5696	.5120	.5178
50	.3557	.3805	.3616	.3636	50	.4975	.5735	.5148	.5207
21° 00′	.3584	.3839	.3645	.3665	**30° 00′**	.5000	.5774	.5176	.5236
10	.3611	.3872	.3673	.3694	10	.5025	.5812	.5204	.5265
20	.3638	.3906	.3702	.3723	20	.5050	.5851	.5233	.5294
30	.3665	.3939	.3730	.3752	30	.5075	.5890	.5261	.5323
40	.3692	.3973	.3759	.3782	40	.5100	.5930	.5289	.5352
50	.3719	.4006	.3788	.3811	50	.5125	.5969	.5317	.5381
22° 00′	.3746	.4040	.3816	.3840	**31° 00′**	.5150	.6009	.5345	.5411
10	.3773	.4074	.3845	.3869	10	.5175	.6048	.5373	.5440
20	.3800	.4108	.3873	.3898	20	.5200	.6088	.5401	.5469
30	.3827	.4142	.3902	.3927	30	.5225	.6128	.5429	.5498
40	.3854	.4176	.3930	.3956	40	.5250	.6168	.5457	.5527
50	.3881	.4210	.3959	.3985	50	.5275	.6208	.5485	.5556
23° 00′	.3907	.4245	.3987	.4014	**32° 00′**	.5299	.6249	.5513	.5585
10	.3934	.4279	.4016	.4043	10	.5324	.6289	.5541	.5614
20	.3961	.4314	.4044	.4072	20	.5348	.6330	.5569	.5643
30	.3987	.4348	.4073	.4102	30	.5373	.6371	.5597	.5672
40	.4014	.4383	.4101	.4131	40	.5398	.6412	.5625	.5701
50	.4041	.4417	.4130	.4160	50	.5422	.6453	.5652	.5730
24° 00′	.4067	.4452	.4158	.4189	**33° 00′**	.5446	.6494	.5680	.5760
10	.4094	.4487	.4187	.4218	10	.5471	.6536	.5708	.5789
20	.4120	.4522	.4215	.4247	20	.5495	.6577	.5736	.5818
30	.4147	.4557	.4244	.4276	30	.5519	.6619	.5764	.5847
40	.4173	.4592	.4272	.4305	40	.5544	.6661	.5792	.5876
50	.4200	.4628	.4300	.4334	50	.5568	.6703	.5820	.5905
25° 00′	.4226	.4663	.4329	.4363	**34° 00′**	.5592	.6745	.5847	.5934
10	.4253	.4699	.4357	.4392	10	.5616	.6787	.5875	.5963
20	.4279	.4734	.4386	.4422	20	.5640	.6830	.5903	.5992
30	.4305	.4770	.4414	.4451	30	.5664	.6873	.5931	.6021
40	.4331	.4806	.4442	.4480	40	.5688	.6916	.5959	.6050
50	.4358	.4841	.4471	.4509	50	.5712	.6959	.5986	.6080
26° 00′	.4384	.4877	.4499	.4538	**35° 00′**	.5736	.7002	.6014	.6109
10	.4410	.4913	.4527	.4567	10	.5760	.7046	.6042	.6138
20	.4436	.4950	.4556	.4596	20	.5783	.7089	.6070	.6167
30	.4462	.4986	.4584	.4625	30	.5807	.7133	.6097	.6196
40	.4488	.5022	.4612	.4654	40	.5831	.7177	.6125	.6225
50	.4514	.5059	.4641	.4683	50	.5854	.7221	.6153	.6254
27° 00′	.4540	.5095	.4669	.4712	**36° 00′**	.5878	.7265	.6180	.6283

Quantities Determined by a Given Angle

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
36° 00′	.5878	.7265	.6180	.6283	**45° 00′**	.7071	1.0000	.7654	.7854
10	.5901	.7310	.6208	.6312	10	.7092	1.0058	.7681	.7883
20	.5925	.7355	.6236	.6341	20	.7112	1.0117	.7707	.7912
30	.5948	.7400	.6263	.6370	30	.7133	1.0176	.7734	.7941
40	.5972	.7445	.6291	.6400	40	.7153	1.0235	.7761	.7970
50	.5995	.7490	.6318	.6429	50	.7173	1.0295	.7788	.7999
37° 00′	.6018	.7536	.6346	.6458	**46° 00′**	.7193	1.0355	.7815	.8029
10	.6041	.7581	.6374	.6487	10	.7214	1.0416	.7841	.8058
20	.6065	.7627	.6401	.6516	20	.7234	1.0477	.7868	.8087
30	.6088	.7673	.6429	.6545	30	.7254	1.0538	.7895	.8116
40	.6111	.7720	.6456	.6574	40	.7274	1.0599	.7922	.8145
50	.6134	.7766	.6484	.6603	50	.7294	1.0661	.7948	.8174
38° 00′	.6157	.7813	.6511	.6632	**47° 00′**	.7314	1.0724	.7975	.8203
10	.6180	.7860	.6539	.6661	10	.7333	1.0786	.8002	.8232
20	.6202	.7907	.6566	.6690	20	.7353	1.0850	.8028	.8261
30	.6225	.7954	.6594	.6720	30	.7373	1.0913	.8055	.8290
40	.6248	.8002	.6621	.6749	40	.7392	1.0977	.8082	.8319
50	.6271	.8050	.6649	.6778	50	.7412	1.1041	.8108	.8348
39° 00′	.6293	.8098	.6676	.6807	**48° 00′**	.7431	1.1106	.8135	.8378
10	.6316	.8146	.6704	.6836	10	.7451	1.1171	.8161	.8407
20	.6338	.8195	.6731	.6865	20	.7470	1.1237	.8188	.8436
30	.6361	.8243	.6758	.6894	30	.7490	1.1303	.8214	.8465
40	.6383	.8292	.6786	.6923	40	.7509	1.1369	.8241	.8494
50	.6406	.8342	.6813	.6952	50	.7528	1.1436	.8267	.8523
40° 00′	.6428	.8391	.6840	.6981	**49° 00′**	.7547	1.1504	.8294	.8552
10	.6450	.8441	.6868	.7010	10	.7566	1.1571	.8320	.8581
20	.6472	.8491	.6895	.7039	20	.7585	1.1640	.8347	.8610
30	.6494	.8541	.6922	.7069	30	.7604	1.1708	.8373	.8639
40	.6517	.8591	.6950	.7098	40	.7623	1.1778	.8400	.8668
50	.6539	.8642	.6977	.7127	50	.7642	1.1847	.8426	.8698
41° 00′	.6561	.8693	.7004	.7156	**50° 00′**	.7660	1.1918	.8452	.8727
10	.6583	.8744	.7031	.7185	10	.7679	1.1988	.8479	.8756
20	.6604	.8796	.7059	.7214	20	.7698	1.2059	.8505	.8785
30	.6626	.8847	.7086	.7243	30	.7716	1.2131	.8531	.8814
40	.6648	.8899	.7113	.7272	40	.7735	1.2203	.8558	.8843
50	.6670	.8952	.7140	.7301	50	.7753	1.2276	.8584	.8872
42° 00′	.6691	.9004	.7167	.7330	**51° 00′**	.7771	1.2349	.8610	.8901
10	.6713	.9057	.7195	.7359	10	.7790	1.2423	.8636	.8930
20	.6734	.9110	.7222	.7389	20	.7808	1.2497	.8663	.8959
30	.6756	.9163	.7249	.7418	30	.7826	1.2572	.8689	.8988
40	.6777	.9217	.7276	.7447	40	.7844	1.2647	.8715	.9018
50	.6799	.9271	.7303	.7476	50	.7862	1.2723	.8741	.9047
43° 00′	.6820	.9325	.7330	.7505	**52° 00′**	.7880	1.2799	.8767	.9076
10	.6841	.9380	.7357	.7534	10	.7898	1.2876	.8794	.9105
20	.6862	.9435	.7384	.7563	20	.7916	1.2954	.8820	.9134
30	.6884	.9490	.7411	.7592	30	.7934	1.3032	.8846	.9163
40	.6905	.9545	.7438	.7621	40	.7951	1.3111	.8872	.9192
50	.6926	.9601	.7465	.7650	50	.7969	1.3190	.8898	.9221
44° 00′	.6947	.9657	.7492	.7679	**53° 00′**	.7986	1.3270	.8924	.9250
10	.6967	.9713	.7519	.7709	10	.8004	1.3351	.8950	.9279
20	.6988	.9770	.7546	.7738	20	.8021	1.3432	.8976	.9308
30	.7009	.9827	.7573	.7767	30	.8039	1.3514	.9002	.9338
40	.7030	.9884	.7600	.7796	40	.8056	1.3597	.9028	.9367
50	.7050	.9942	.7627	.7825	50	.8073	1.3680	.9054	.9396
45° 00′	.7071	1.0000	.7654	.7854	**54° 00′**	.8090	1.3764	.9080	.9425

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
54° 00′	.8090	1.3764	.9080	.9425	**63° 00′**	.8910	1.9626	1.0450	1.0996
10	.8107	1.3848	.9106	.9454	10	.8923	1.9768	1.0475	1.1025
20	.8124	1.3934	.9132	.9483	20	.8936	1.9912	1.0500	1.1054
30	.8141	1.4019	.9157	.9512	30	.8949	2.0057	1.0524	1.1083
40	.8158	1.4106	.9183	.9541	40	.8962	2.0204	1.0549	1.1112
50	.8175	1.4193	.9209	.9570	50	.8975	2.0353	1.0574	1.1141
55° 00′	.8192	1.4281	.9235	.9599	**64° 00′**	.8988	2.0503	1.0598	1.1170
10	.8208	1.4370	.9261	.9628	10	.9001	2.0655	1.0623	1.1199
20	.8225	1.4460	.9287	.9657	20	.9013	2.0809	1.0648	1.1228
30	.8241	1.4550	.9312	.9687	30	.9026	2.0965	1.0672	1.1257
40	.8258	1.4641	.9338	.9716	40	.9038	2.1123	1.0697	1.1286
50	.8274	1.4733	.9364	.9745	50	.9051	2.1283	1.0721	1.1316
56° 00′	.8290	1.4826	.9389	.9774	**65° 00′**	.9063	2.1445	1.0746	1.1345
10	.8307	1.4919	.9415	.9803	10	.9075	2.1609	1.0771	1.1374
20	.8323	1.5013	.9441	.9832	20	.9088	2.1775	1.0795	1.1403
30	.8339	1.5108	.9466	.9861	30	.9100	2.1943	1.0820	1.1432
40	.8355	1.5204	.9492	.9890	40	.9112	2.2113	1.0844	1.1461
50	.8371	1.5301	.9518	.9919	50	.9124	2.2286	1.0868	1.1490
57° 00′	.8387	1.5399	.9543	.9948	**66° 00′**	.9135	2.2460	1.0893	1.1519
10	.8403	1.5497	.9569	.9977	10	.9147	2.2637	1.0917	1.1548
20	.8418	1.5597	.9594	1.0007	20	.9159	2.2817	1.0942	1.1577
30	.8434	1.5697	.9620	1.0036	30	.9171	2.2998	1.0966	1.1606
40	.8450	1.5798	.9645	1.0065	40	.9182	2.3183	1.0990	1.1636
50	.8465	1.5900	.9671	1.0094	50	.9194	2.3369	1.1014	1.1665
58° 00′	.8480	1.6003	.9696	1.0123	**67° 00′**	.9205	2.3559	1.1039	1.1694
10	.8496	1.6107	.9722	1.0152	10	.9216	2.3750	1.1063	1.1723
20	.8511	1.6212	.9747	1.0181	20	.9228	2.3945	1.1087	1.1752
30	.8526	1.6319	.9772	1.0210	30	.9239	2.4142	1.1111	1.1781
40	.8542	1.6426	.9798	1.0239	40	.9250	2.4342	1.1136	1.1810
50	.8557	1.6534	.9823	1.0268	50	.9261	2.4545	1.1160	1.1839
59° 00′	.8572	1.6643	.9848	1.0297	**68° 00′**	.9272	2.4751	1.1184	1.1868
10	.8587	1.6753	.9874	1.0327	10	.9283	2.4960	1.1208	1.1897
20	.8601	1.6864	.9899	1.0356	20	.9293	2.5172	1.1232	1.1926
30	.8616	1.6977	.9924	1.0385	30	.9304	2.5386	1.1256	1.1956
40	.8631	1.7090	.9950	1.0414	40	.9315	2.5605	1.1280	1.1985
50	.8646	1.7205	.9975	1.0443	50	.9325	2.5826	1.1304	1.2014
60° 00′	.8660	1.7321	1.0000	1.0472	**69° 00′**	.9336	2.6051	1.1328	1.2043
10	.8675	1.7437	1.0025	1.0501	10	.9346	2.6279	1.1352	1.2072
20	.8689	1.7556	1.0050	1.0530	20	.9356	2.6511	1.1376	1.2101
30	.8704	1.7675	1.0075	1.0559	30	.9367	2.6746	1.1400	1.2130
40	.8718	1.7796	1.0101	1.0588	40	.9377	2.6985	1.1424	1.2159
50	.8732	1.7917	1.0126	1.0617	50	.9387	2.7228	1.1448	1.2188
61° 00′	.8746	1.8040	1.0151	1.0647	**70° 00′**	.9397	2.7475	1.1472	1.2217
10	.8760	1.8165	1.0176	1.0676	10	.9407	2.7725	1.1495	1.2246
20	.8774	1.8291	1.0201	1.0705	20	.9417	2.7980	1.1519	1.2275
30	.8788	1.8418	1.0226	1.0734	30	.9426	2.8239	1.1543	1.2305
40	.8802	1.8546	1.0251	1.0763	40	.9436	2.8502	1.1567	1.2334
50	.8816	1.8676	1.0276	1.0792	50	.9446	2.8770	1.1590	1.2363
62° 00′	.8829	1.8807	1.0301	1.0821	**71° 00′**	.9455	2.9042	1.1614	1.2392
10	.8843	1.8940	1.0326	1.0850	10	.9465	2.9319	1.1638	1.2421
20	.8857	1.9074	1.0351	1.0879	20	.9474	2.9600	1.1661	1.2450
30	.8870	1.9210	1.0375	1.0908	30	.9483	2.9887	1.1685	1.2479
40	.8884	1.9347	1.0400	1.0937	40	.9492	3.0178	1.1709	1.2508
50	.8897	1.9486	1.0425	1.0966	50	.9502	3.0475	1.1732	1.2537
63° 00′	.8910	1.9626	1.0450	1.0996	**72° 00′**	.9511	3.0777	1.1756	1.2566

Angle	Sine	Tangent	Chord	Arc	Angle	Sine	Tangent	Chord	Arc
72° 00′	.9511	3.0777	1.1756	1.2566	**81° 00′**	.9877	6.3138	1.2989	1.4137
10	.9520	3.1084	1.1779	1.2595	10	.9881	6.4348	1.3011	1.4166
20	.9528	3.1397	1.1803	1.2625	20	.9886	6.5606	1.3033	1.4195
30	.9537	3.1716	1.1826	1.2654	30	.9890	6.6912	1.3055	1.4224
40	.9546	3.2041	1.1850	1.2683	40	.9894	6.8269	1.3077	1.4254
50	.9555	3.2371	1.1873	1.2712	50	.9899	6.9682	1.3099	1.4283
73° 00′	.9563	3.2709	1.1896	1.2741	**82° 00′**	.9903	7.1154	1.3121	1.4312
10	.9572	3.3052	1.1920	1.2770	10	.9907	7.2687	1.3143	1.4341
20	.9580	3.3402	1.1943	1.2799	20	.9911	7.4287	1.3165	1.4370
30	.9588	3.3759	1.1966	1.2828	30	.9914	7.5958	1.3187	1.4399
40	.9596	3.4124	1.1990	1.2857	40	.9918	7.7704	1.3209	1.4428
50	.9605	3.4495	1.2013	1.2886	50	.9922	7.9530	1.3231	1.4457
74° 00′	.9613	3.4874	1.2036	1.2915	**83° 00′**	.9925	8.1443	1.3252	1.4486
10	.9621	3.5261	1.2060	1.2945	10	.9929	8.3450	1.3274	1.4515
20	.9628	3.5656	1.2083	1.2974	20	.9932	8.5555	1.3296	1.4544
30	.9636	3.6059	1.2106	1.3003	30	.9936	8.7769	1.3318	1.4573
40	.9644	3.6470	1.2129	1.3032	40	.9939	9.0098	1.3339	1.4603
50	.9652	3.6891	1.2152	1.3061	50	.9942	9.2553	1.3361	1.4632
75° 00′	.9659	3.7321	1.2175	1.3090	**84° 00′**	.9945	9.5144	1.3383	1.4661
10	.9667	3.7760	1.2198	1.3119	10	.9948	9.7882	1.3404	1.4690
20	.9674	3.8208	1.2221	1.3148	20	.9951	10.078	1.3426	1.4719
30	.9681	3.8667	1.2244	1.3177	30	.9954	10.385	1.3447	1.4748
40	.9689	3.9136	1.2267	1.3206	40	.9957	10.712	1.3469	1.4777
50	.9696	3.9617	1.2290	1.3235	50	.9959	11.059	1.3490	1.4806
76° 00′	.9703	4.0108	1.2313	1.3265	**85° 00′**	.9962	11.430	1.3512	1.4835
10	.9710	4.0611	1.2336	1.3294	10	.9964	11.826	1.3533	1.4864
20	.9717	4.1126	1.2359	1.3323	20	.9967	12.251	1.3555	1.4893
30	.9724	4.1653	1.2382	1.3352	30	.9969	12.706	1.3576	1.4923
40	.9730	4.2193	1.2405	1.3381	40	.9971	13.197	1.3597	1.4952
50	.9737	4.2747	1.2428	1.3410	50	.9974	13.727	1.3619	1.4981
77° 00′	.9744	4.3315	1.2450	1.3439	**86° 00′**	.9976	14.301	1.3640	1.5010
10	.9750	4.3897	1.2473	1.3468	10	.9978	14.924	1.3661	1.5039
20	.9757	4.4494	1.2496	1.3497	20	.9980	15.605	1.3682	1.5068
30	.9763	4.5107	1.2518	1.3526	30	.9981	16.350	1.3704	1.5097
40	.9769	4.5736	1.2541	1.3555	40	.9983	17.169	1.3725	1.5126
50	.9775	4.6382	1.2564	1.3584	50	.9985	18.075	1.3746	1.5155
78° 00′	.9781	4.7046	1.2586	1.3614	**87° 00′**	.9986	19.081	1.3767	1.5184
10	.9787	4.7729	1.2609	1.3643	10	.9988	20.206	1.3788	1.5213
20	.9793	4.8430	1.2632	1.3672	20	.9989	21.470	1.3809	1.5243
30	.9799	4.9152	1.2654	1.3701	30	.9990	22.904	1.3830	1.5272
40	.9805	4.9894	1.2677	1.3730	40	.9992	24.542	1.3851	1.5301
50	.9811	5.0658	1.2699	1.3759	50	.9993	26.432	1.3872	1.5330
79° 00′	.9816	5.1446	1.2722	1.3788	**88° 00′**	.9994	28.636	1.3893	1.5359
10	.9822	5.2257	1.2744	1.3817	10	.9995	31.242	1.3914	1.5388
20	.9827	5.3093	1.2766	1.3846	20	.9996	34.368	1.3935	1.5417
30	.9833	5.3955	1.2789	1.3875	30	.9997	38.188	1.3956	1.5446
40	.9838	5.4845	1.2811	1.3904	40	.9997	42.964	1.3977	1.5475
50	.9843	5.5764	1.2833	1.3934	50	.9998	49.104	1.3997	1.5504
80° 00′	.9848	5.6713	1.2856	1.3963	**89° 00′**	.9998	57.290	1.4018	1.5533
10	.9853	5.7694	1.2878	1.3992	10	.9999	68.750	1.4039	1.5563
20	.9858	5.8708	1.2900	1.4021	20	.9999	85.940	1.4060	1.5592
30	.9863	5.9758	1.2922	1.4050	30	1.0000	114.59	1.4080	1.5621
40	.9868	6.0844	1.2945	1.4079	40	1.0000	171.89	1.4101	1.5650
50	.9872	6.1970	1.2967	1.4108	50	1.0000	343.77	1.4122	1.5679
81° 00′	.9877	6.3138	1.2989	1.4137	**90° 00′**	1.0000	——	1.4142	1.5708

TABLE II — POWERS AND ROOTS

EXPLANATION OF TABLE II

1. Squares and Cubes. The *squares* of numbers between 1.00 and 10.00 at intervals of .01 are given in column headed n^2. To find the square of any other number, divide (or multiply) the given number by 10 to reduce it to a number between 1 and 10; find the square of this last number; multiply (or divide) the square thus found by 10 twice as many times as you did the given number.

The *cube* is given in the column headed n^3. To find the cube of any number not between 1 and 10, first reduce that number to a number between 1 and 10 by dividing (or multiplying) by a power of 10. Multiply (or divide) the result found by three times as high a power of 10 as was used to reduce the given number.

2. Square Roots. The *square roots* of numbers between 1 and 10 are found in the column headed $\sqrt{n}$.

The square roots of numbers between 10 and 100 may be found in the column headed $\sqrt{10n}$.

The square roots of numbers between 100 and 1000 may be found in the column headed $\sqrt{n}$ by multiplying the given root by 10, since $\sqrt{100n} = 10\sqrt{n}$.

Other square roots may be found in a similar manner.

3. Cube Roots. The column headed:

$\sqrt[3]{n}$ gives cube roots of numbers between 1 and 10;
$\sqrt[3]{10n}$ gives cube roots of numbers between 10 and 100;
$\sqrt[3]{100n}$ gives cube roots of numbers between 100 and 1000.

To find the cube root of a number between 1000 and 10000, take 10 times the value found in the column headed $\sqrt[3]{n}$, since $\sqrt[3]{1000n} = 10\sqrt[3]{n}$.

Other cube roots may be found similarly.

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
1.00	1.0000	1.00000	3.16228	1.00000	1.00000	2.15443	4.64159
1.01	1.0201	1.00499	3.17805	1.03030	1.00332	2.16159	4.65701
1.02	1.0404	1.00995	3.19374	1.06121	1.00662	2.16870	4.67233
1.03	1.0609	1.01489	3.20936	1.09273	1.00990	2.17577	4.68755
1.04	1.0816	1.01980	3.22490	1.12486	1.01316	2.18279	4.70267
1.05	1.1025	1.02470	3.24037	1.15762	1.01640	2.18976	4.71769
1.06	1.1236	1.02956	3.25576	1.19102	1.01961	2.19669	4.73262
1.07	1.1449	1.03441	3.27109	1.22504	1.02281	2.20358	4.74746
1.08	1.1664	1.03923	3.28634	1.25971	1.02599	2.21042	4.76220
1.09	1.1881	1.04403	3.30151	1.29503	1.02914	2.21722	4.77686
1.10	1.2100	1.04881	3.31662	1.33100	1.03228	2.22398	4.79142
1.11	1.2321	1.05357	3.33167	1.36763	1.03540	2.23070	4.80590
1.12	1.2544	1.05830	3.34664	1.40493	1.03850	2.23738	4.82028
1.13	1.2769	1.06301	3.36155	1.44290	1.04158	2.24402	4.83459
1.14	1.2996	1.06771	3.37639	1.48154	1.04464	2.25062	4.84881
1.15	1.3225	1.07238	3.39116	1.52088	1.04769	2.25718	4.86294
1.16	1.3456	1.07703	3.40588	1.56090	1.05072	2.26370	4.87700
1.17	1.3689	1.08167	3.42053	1.60161	1.05373	2.27019	4.89097
1.18	1.3924	1.08628	3.43511	1.64303	1.05672	2.27664	4.90487
1.19	1.4161	1.09087	3.44964	1.68516	1.05970	2.28305	4.91868
1.20	1.4400	1.09545	3.46410	1.72800	1.06266	2.28943	4.93242
1.21	1.4641	1.10000	3.47851	1.77156	1.06560	2.29577	4.94609
1.22	1.4884	1.10454	3.49285	1.81585	1.06853	2.30208	4.95968
1.23	1.5129	1.10905	3.50714	1.86087	1.07144	2.30835	4.97319
1.24	1.5376	1.11355	3.52136	1.90662	1.07434	2.31459	4.98663
1.25	1.5625	1.11803	3.53553	1.95312	1.07722	2.32079	5.00000
1.26	1.5876	1.12250	3.54965	2.00038	1.08008	2.32697	5.01330
1.27	1.6129	1.12694	3.56371	2.04838	1.08293	2.33311	5.02653
1.28	1.6384	1.13137	3.57771	2.09715	1.08577	2.33921	5.03968
1.29	1.6641	1.13578	3.59166	2.14669	1.08859	2.34529	5.05277
1.30	1.6900	1.14018	3.60555	2.19700	1.09139	2.35133	5.06580
1.31	1.7161	1.14455	3.61939	2.24809	1.09418	2.35735	5.07875
1.32	1.7424	1.14891	3.63318	2.29997	1.09696	2.36333	5.09164
1.33	1.7689	1.15326	3.64692	2.35264	1.09972	2.36928	5.10447
1.34	1.7956	1.15758	3.66060	2.40610	1.10247	2.37521	5.11723
1.35	1.8225	1.16190	3.67423	2.46038	1.10521	2.38110	5.12993
1.36	1.8496	1.16619	3.68782	2.51546	1.10793	2.38697	5.14256
1.37	1.8769	1.17047	3.70135	2.57135	1.11064	2.39280	5.15514
1.38	1.9044	1.17473	3.71484	2.62807	1.11334	2.39861	5.16765
1.39	1.9321	1.17898	3.72827	2.68562	1.11602	2.40439	5.18010
1.40	1.9600	1.18322	3.74166	2.74400	1.11869	2.41014	5.19249
1.41	1.9881	1.18743	3.75500	2.80322	1.12135	2.41587	5.20483
1.42	2.0164	1.19164	3.76829	2.86329	1.12399	2.42156	5.21710
1.43	2.0449	1.19583	3.78153	2.92421	1.12662	2.42724	5.22932
1.44	2.0736	1.20000	3.79473	2.98598	1.12924	2.43288	5.24148
1.45	2.1025	1.20416	3.80789	3.04862	1.13185	2.43850	5.25359
1.46	2.1316	1.20830	3.82099	3.11214	1.13445	2.44409	5.26564
1.47	2.1609	1.21244	3.83406	3.17652	1.13703	2.44966	5.27763
1.48	2.1904	1.21655	3.84708	3.24179	1.13960	2.45520	5.28957
1.49	2.2201	1.22066	3.86005	3.30795	1.14216	2.46072	5.30146

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
1.50	2.2500	1.22474	3.87298	3.37500	1.14471	2.46621	5.31329
1.51	2.2801	1.22882	3.88587	3.44295	1.14725	2.47168	5.32507
1.52	2.3104	1.23288	3.89872	3.51181	1.14978	2.47712	5.33680
1.53	2.3409	1.23693	3.91152	3.58158	1.15230	2.48255	5.34848
1.54	2.3716	1.24097	3.92428	3.65226	1.15480	2.48794	5.36011
1.55	2.4025	1.24499	3.93700	3.72388	1.15729	2.49332	5.37169
1.56	2.4336	1.24900	3.94968	3.79642	1.15978	2.49867	5.38321
1.57	2.4649	1.25300	3.96232	3.86989	1.16225	2.50399	5.39469
1.58	2.4964	1.25698	3.97492	3.94431	1.16471	2.50930	5.40612
1.59	2.5281	1.26095	3.98748	4.01968	1.16717	2.51458	5.41750
1.60	2.5600	1.26491	4.00000	4.09600	1.16961	2.51984	5.42884
1.61	2.5921	1.26886	4.01248	4.17328	1.17204	2.52508	5.44012
1.62	2.6244	1.27279	4.02492	4.25153	1.17446	2.53030	5.45136
1.63	2.6569	1.27671	4.03733	4.33075	1.17687	2.53549	5.46256
1.64	2.6896	1.28062	4.04969	4.41094	1.17927	2.54067	5.47370
1.65	2.7225	1.28452	4.06202	4.49212	1.18167	2.54582	5.48481
1.66	2.7556	1.28841	4.07431	4.57430	1.18405	2.55095	5.49586
1.67	2.7889	1.29228	4.08656	4.65746	1.18642	2.55607	5.50688
1.68	2.8224	1.29615	4.09878	4.74163	1.18878	2.56116	5.51785
1.69	2.8561	1.30000	4.11096	4.82681	1.19114	2.56623	5.52877
1.70	2.8900	1.30384	4.12311	4.91300	1.19348	2.57128	5.53966
1.71	2.9241	1.30767	4.13521	5.00021	1.19582	2.57631	5.55050
1.72	2.9584	1.31149	4.14729	5.08845	1.19815	2.58133	5.56130
1.73	2.9929	1.31529	4.15933	5.17772	1.20046	2.58632	5.57205
1.74	3.0276	1.31909	4.17133	5.26802	1.20277	2.59129	5.58277
1.75	3.0625	1.32288	4.18330	5.35938	1.20507	2.59625	5.59344
1.76	3.0976	1.32665	4.19524	5.45178	1.20736	2.60118	5.60408
1.77	3.1329	1.33041	4.20714	5.54523	1.20964	2.60610	5.61467
1.78	3.1684	1.33417	4.21900	5.63975	1.21192	2.61100	5.62523
1.79	3.2041	1.33791	4.23084	5.73534	1.21418	2.61588	5.63574
1.80	3.2400	1.34164	4.24264	5.83200	1.21644	2.62074	5.64622
1.81	3.2761	1.34536	4.25441	5.92974	1.21869	2.62559	5.65665
1.82	3.3124	1.34907	4.26615	6.02857	1.22093	2.63041	5.66705
1.83	3.3489	1.35277	4.27785	6.12849	1.22316	2.63522	5.67741
1.84	3.3856	1.35647	4.28952	6.22950	1.22539	2.64001	5.68773
1.85	3.4225	1.36015	4.30116	6.33162	1.22760	2.64479	5.69802
1.86	3.4596	1.36382	4.31277	6.43486	1.22981	2.64954	5.70827
1.87	3.4969	1.36748	4.32435	6.53920	1.23201	2.65428	5.71848
1.88	3.5344	1.37113	4.33590	6.64467	1.23420	2.65901	5.72865
1.89	3.5721	1.37477	4.34741	6.75127	1.23639	2.66371	5.73879
1.90	3.6100	1.37840	4.35890	6.85900	1.23856	2.66840	5.74890
1.91	3.6481	1.38203	4.37035	6.96787	1.24073	2.67307	5.75897
1.92	3.6864	1.38564	4.38178	7.07789	1.24289	2.67773	5.76900
1.93	3.7249	1.38924	4.39318	7.18906	1.24505	2.68237	5.77900
1.94	3.7636	1.39284	4.40454	7.30138	1.24719	2.68700	5.78896
1.95	3.8025	1.39642	4.41588	7.41488	1.24933	2.69161	5.79889
1.96	3.8416	1.40000	4.42719	7.52954	1.25146	2.69620	5.80879
1.97	3.8809	1.40357	4.43847	7.64537	1.25359	2.70078	5.81865
1.98	3.9204	1.40712	4.44972	7.76239	1.25571	2.70534	5.82848
1.99	3.9601	1.41067	4.46094	7.88060	1.25782	2.70989	5.83827

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
2.00	4.0000	1.41421	4.47214	8.00000	1.25992	2.71442	5.84804
2.01	4.0401	1.41774	4.48330	8.12060	1.26202	2.71893	5.85777
2.02	4.0804	1.42127	4.49444	8.24241	1.26411	2.72344	5.86746
2.03	4.1209	1.42478	4.50555	8.36543	1.26619	2.72792	5.87713
2.04	4.1616	1.42829	4.51664	8.48966	1.26827	2.73239	5.88677
2.05	4.2025	1.43178	4.52769	8.61512	1.27033	2.73685	5.89637
2.06	4.2436	1.43527	4.53872	8.74182	1.27240	2.74129	5.90594
2.07	4.2849	1.43875	4.54973	8.86974	1.27445	2.74572	5.91548
2.08	4.3264	1.44222	4.56070	8.99891	1.27650	2.75014	5.92499
2.09	4.3681	1.44568	4.57165	9.12933	1.27854	2.75454	5.93447
2.10	4.4100	1.44914	4.58258	9 26100	1.28058	2.75892	5.94392
2.11	4.4521	1.45258	4.59347	9.39393	1.28261	2.76330	5.95334
2.12	4.4944	1.45602	4.60435	9.52813	1.28463	2.76766	5.96273
2.13	4.5369	1.45945	4.61519	9.66360	1.28665	2.77200	5.97209
2.14	4.5796	1.46287	4.62601	9.80034	1.28866	2.77633	5.98142
2.15	4.6225	1.46629	4.63681	9.93838	1.29066	2.78065	5.99073
2.16	4.6656	1.46969	4.64758	10.0777	1.29266	2.78495	6.00000
2.17	4.7089	1.47309	4.65833	10.2183	1.29465	2.78924	6.00925
2.18	4.7524	1.47648	4.66905	10.3602	1.29664	2.79352	6.01846
2.19	4.7961	1.47986	4.67974	10.5035	1.29862	2.79779	6.02765
2.20	4.8400	1.48324	4.69042	10.6480	1.30059	2.80204	6.03681
2.21	4.8841	1.48661	4.70106	10.7939	1.30256	2.80628	6.04594
2.22	4.9284	1.48997	4.71169	10.9410	1.30452	2.81050	6.05505
2.23	4.9729	1.49332	4.72229	11.0896	1.30648	2.81472	6.06413
2.24	5.0176	1.49666	4.73286	11.2394	1.30843	2.81892	6.07318
2.25	5.0625	1.50000	4.74342	11.3906	1.31037	2.82311	6.08220
2.26	5.1076	1.50333	4.75395	11.5432	1.31231	2.82728	6.09120
2.27	5.1529	1.50665	4.76445	11.6971	1.31424	2.83145	6.10017
2.28	5.1984	1.50997	4.77493	11.8524	1.31617	2.83560	6.10911
2.29	5.2441	1.51327	4.78539	12.0090	1.31809	2.83974	6.11803
2.30	5.2900	1.51658	4.79583	12.1670	1.32001	2.84387	6.12693
2.31	5.3361	1.51987	4.80625	12.3264	1.32192	2.84798	6.13579
2.32	5.3824	1.52315	4.81664	12.4872	1.32382	2.85209	6.14463
2.33	5.4289	1.52643	4.82701	12.6493	1.32572	2.85618	6.15345
2.34	5.4756	1.52971	4.83735	12.8129	1.32761	2.86026	6.16224
2.35	5.5225	1.53297	4.84768	12.9779	1.32950	2.86433	6.17101
2.36	5.5696	1.53623	4.85798	13.1443	1.33139	2.86838	6.17975
2.37	5.6169	1.53948	4.86826	13.3121	1.33326	2.87243	6.18846
2.38	5.6644	1.54272	4.87852	13.4813	1.33514	2.87646	6.19715
2.39	5.7121	1.54596	4.88876	13.6519	1.33700	2.88049	6.20582
2.40	5.7600	1.54919	4.89898	13.8240	1.33887	2.88450	6.21447
2.41	5.8081	1.55242	4.90918	13.9975	1.34072	2.88850	6.22308
2.42	5.8564	1.55563	4.91935	14.1725	1.34257	2.89249	6.23168
2.43	5.9049	1.55885	4.92950	14.3489	1.34442	2.89647	6.24025
2.44	5.9536	1.56205	4.93964	14.5268	1.34626	2.90044	6.24880
2.45	6.0025	1.56525	4.94975	14.7061	1.34810	2.90439	6.25732
2.46	6.0516	1.56844	4.95984	14.8869	1.34993	2.90834	6.26583
2.47	6.1009	1.57162	4.96991	15.0692	1.35176	2.91227	6.27431
2.48	6.1504	1.57480	4.97996	15.2530	1.35358	2.91620	6.28276
2.49	6.2001	1.57797	4.98999	15.4382	1.35540	2.92011	6.29119

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
2.50	6.2500	1.58114	5.00000	15.6250	1.35721	2.92402	6.29961
2.51	6.3001	1.58430	5.00999	15.8133	1.35902	2.92791	6.30799
2.52	6.3504	1.58745	5.01996	16.0030	1.36082	2.93179	6.31636
2.53	6.4009	1.59060	5.02991	16.1943	1.36262	2.93567	6.32470
2.54	6.4516	1.59374	5.03984	16.3871	1.36441	2.93953	6.33303
2.55	6.5025	1.59687	5.04975	16.5814	1.36620	2.94338	6.34133
2.56	6.5536	1.60000	5.05964	16.7772	1.36798	2.94723	6.34960
2.57	6.6049	1.60312	5.06952	16.9746	1.36976	2.95106	6.35786
2.58	6.6564	1.60624	5.07937	17.1735	1.37153	2.95488	6.36610
2.59	6.7081	1.60935	5.08920	17.3740	1.37330	2.95869	6.37431
2.60	6.7600	1.61245	5.09902	17.5760	1.37507	2.96250	6.38250
2.61	6.8121	1.61555	5.10882	17.7796	1.37683	2.96629	6.39068
2.62	6.8644	1.61864	5.11859	17.9847	1.37859	2.97007	6.39883
2.63	6.9169	1.62173	5.12835	18.1914	1.38034	2.97385	6.40696
2.64	6.9696	1.62481	5.13809	18.3997	1.38208	2.97761	6.41507
2.65	7.0225	1.62788	5.14782	18.6096	1.38383	2.98137	6.42316
2.66	7.0756	1.63095	5.15752	18.8211	1.38557	2.98511	6.43123
2.67	7.1289	1.63401	5.16720	19.0342	1.38730	2.98885	6.43928
2.68	7.1824	1.63707	5.17687	19.2488	1.38903	2.99257	6.44731
2.69	7.2361	1.64012	5.18652	19.4651	1.39076	2.99629	6.45531
2.70	7.2900	1.64317	5.19615	19.6830	1.39248	3.00000	6.46330
2.71	7.3441	1.64621	5.20577	19.9025	1.39419	3.00370	6.47127
2.72	7.3984	1.64924	5.21536	20.1236	1.39591	3.00739	6.47922
2.73	7.4529	1.65227	5.22494	20.3464	1.39761	3.01107	6.48715
2.74	7.5076	1.65529	5.23450	20.5708	1.39932	3.01474	6.49507
2.75	7.5625	1.65831	5.24404	20.7969	1.40102	3.01841	6.50296
2.76	7.6176	1.66132	5.25357	21.0246	1.40272	3.02206	6.51083
2.77	7.6729	1.66433	5.26308	21.2539	1.40441	3.02570	6.51868
2.78	7.7284	1.66733	5.27257	21.4850	1.40610	3.02934	6.52652
2.79	7.7841	1.67033	5.28205	21.7176	1.40778	3.03297	6.53434
2.80	7.8400	1.67332	5.29150	21.9520	1.40946	3.03659	6.54213
2.81	7.8961	1.67631	5.30094	22.1880	1.41114	3.04020	6.54991
2.82	7.9524	1.67929	5.31037	22.4258	1.41281	3.04380	6.55767
2.83	8.0089	1.68226	5.31977	22.6652	1.41448	3.04740	6.56541
2.84	8.0656	1.68523	5.32917	22.9063	1.41614	3.05098	6.57314
2.85	8.1225	1.68819	5.33854	23.1491	1.41780	3.05456	6.58084
2.86	8.1796	1.69115	5.34790	23.3937	1.41946	3.05813	6.58853
2.87	8.2369	1.69411	5.35724	23.6399	1.42111	3.06169	6.59620
2.88	8.2944	1.69706	5.36656	23.8879	1.42276	3.06524	6.60385
2.89	8.3521	1.70000	5.37587	24.1376	1.42440	3.06878	6.61149
2.90	8.4100	1.70294	5.38516	24.3890	1.42604	3.07232	6.61911
2.91	8.4681	1.70587	5.39444	24.6422	1.42768	3.07584	6.62671
2.92	8.5264	1.70880	5.40370	24.8971	1.42931	3.07936	6.63429
2.93	8.5849	1.71172	5.41295	25.1538	1.43094	3.08287	6.64185
2.94	8.6436	1.71464	5.42218	25.4122	1.43257	3.08638	6.64940
2.95	8.7025	1.71756	5.43139	25.6724	1.43419	3.08987	6.65693
2.96	8.7616	1.72047	5.44059	25.9343	1.43581	3.09336	6.66444
2.97	8.8209	1.72337	5.44977	26.1981	1.43743	3.09684	6.67194
2.98	8.8804	1.72627	5.45894	26.4636	1.43904	3.10031	6.67942
2.99	8.9401	1.72916	5.46809	26.7309	1.44065	3.10378	6.68688

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
3.00	9.0000	1.73205	5.47723	27.0000	1.44225	3.10723	6.69433
3.01	9.0601	1.73494	5.48635	27.2709	1.44385	3.11068	6.70176
3.02	9.1204	1.73781	5.49545	27.5436	1.44545	3.11412	6.70917
3.03	9.1809	1.74069	5.50454	27.8181	1.44704	3.11756	6.71657
3.04	9.2416	1.74356	5.51362	28.0945	1.44863	3.12098	6.72395
3.05	9.3025	1.74642	5.52268	28.3726	1.45022	3.12440	6.73132
3.06	9.3636	1.74929	5.53173	28.6526	1.45180	3.12781	6.73866
3.07	9.4249	1.75214	5.54076	28.9344	1.45338	3.13121	6.74600
3.08	9.4864	1.75499	5.54977	29.2181	1.45496	3.13461	6.75331
3.09	9.5481	1.75784	5.55878	29.5036	1.45653	3.13800	6.76061
3.10	9.6100	1.76068	5.56776	29.7910	1.45810	3.14138	6.76790
3.11	9.6721	1.76352	5.57674	30.0802	1.45967	3.14475	6.77517
3.12	9.7344	1.76635	5.58570	30.3713	1.46123	3.14812	6.78242
3.13	9.7969	1.76918	5.59464	30.6643	1.46279	3.15148	6.78966
3.14	9.8596	1.77200	5.60357	30.9591	1.46434	3.15483	6.79688
3.15	9.9225	1.77482	5.61249	31.2559	1.46590	3.15818	6.80409
3.16	9.9856	1.77764	5.62139	31.5545	1.46745	3.16152	6.81128
3.17	10.0489	1.78045	5.63028	31.8550	1.46899	3.16485	6.81846
3.18	10.1124	1.78326	5.63915	32.1574	1.47054	3.16817	6.82562
3.19	10.1761	1.78606	5.64801	32.4618	1.47208	3.17149	6.83277
3.20	10.2400	1.78885	5.65685	32.7680	1.47361	3.17480	6.83990
3.21	10.3041	1.79165	5.66569	33.0762	1.47515	3.17811	6.84702
3.22	10.3684	1.79444	5.67450	33.3862	1.47668	3.18140	6.85412
3.23	10.4329	1.79722	5.68331	33.6983	1.47820	3.18469	6.86121
3.24	10.4976	1.80000	5.69210	34.0122	1.47973	3.18798	6.86829
3.25	10.5625	1.80278	5.70088	34.3281	1.48125	3.19125	6.87534
3.26	10.6276	1.80555	5.70964	34.6460	1.48277	3.19452	6.88239
3.27	10.6929	1.80831	5.71839	34.9658	1.48428	3.19778	6.88942
3.28	10.7584	1.81108	5.72713	35.2876	1.48579	3.20104	6.89643
3.29	10.8241	1.81384	5.73585	35.6113	1.48730	3.20429	6.90344
3.30	10.8900	1.81659	5.74456	35.9370	1.48881	3.20753	6.91042
3.31	10.9561	1.81934	5.75326	36.2647	1.49031	3.21077	6.91740
3.32	11.0224	1.82209	5.76194	36.5944	1.49181	3.21400	6.92436
3.33	11.0889	1.82483	5.77062	36.9260	1.49330	3.21722	6.93130
3.34	11.1556	1.82757	5.77927	37.2597	1.49480	3.22044	6.93823
3.35	11.2225	1.83030	5.78792	37.5954	1.49629	3.22365	6.94515
3.36	11.2896	1.83303	5.79655	37.9331	1.49777	3.22686	6.95205
3.37	11.3569	1.83576	5.80517	38.2728	1.49926	3.23006	6.95894
3.38	11.4244	1.83848	5.81378	38.6145	1.50074	3.23325	6.96582
3.39	11.4921	1.84120	5.82237	38.9582	1.50222	3.23643	6.97268
3.40	11.5600	1.84391	5.83095	39.3040	1.50369	3.23961	6.97953
3.41	11.6281	1.84662	5.83952	39.6518	1.50517	3.24278	6.98637
3.42	11.6964	1.84932	5.84808	40.0017	1.50664	3.24595	6.99319
3.43	11.7649	1.85203	5.85662	40.3536	1.50810	3.24911	7.00000
3.44	11.8336	1.85472	5.86515	40.7076	1.50957	3.25227	7.00680
3.45	11.9025	1.85742	5.87367	41.0636	1.51103	3.25542	7.01358
3.46	11.9716	1.86011	5.88218	41.4217	1.51249	3.25856	7.02035
3.47	12.0409	1.86279	5.89067	41.7819	1.51394	3.26169	7.02711
3.48	12.1104	1.86548	5.89915	42.1442	1.51540	3.26482	7.03385
3.49	12.1801	1.86815	5.90762	42.5085	1.51685	3.26795	7.04058

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
3.50	12.2500	1.87083	5.91608	42.8750	1.51829	3.27107	7.04730
3.51	12.3201	1.87350	5.92453	43.2436	1.51974	3.27418	7.05400
3.52	12.3904	1.87617	5.93296	43.6142	1.52118	3.27729	7.06070
3.53	12.4609	1.87883	5.94138	43.9870	1.52262	3.28039	7.06738
3.54	12.5316	1.88149	5.94979	44.3619	1.52406	3.28348	7.07404
3.55	12.6025	1.88414	5.95819	44.7389	1.52549	3.28657	7.08070
3.56	12.6736	1.88680	5.96657	45.1180	1.52692	3.28965	7.08734
3.57	12.7449	1.88944	5.97495	45.4993	1.52835	3.29273	7.09397
3.58	12.8164	1.89209	5.98331	45.8827	1.52978	3.29580	7.10059
3.59	12.8881	1.89473	5.99166	46.2683	1.53120	3.29887	7.10719
3.60	12.9600	1.89737	6.00000	46.6560	1.53262	3.30193	7.11379
3.61	13.0321	1.90000	6.00833	47.0459	1.53404	3.30498	7.12037
3.62	13.1044	1.90263	6.01664	47.4379	1.53545	3.30803	7.12694
3.63	13.1769	1.90526	6.02495	47.8321	1.53686	3.31107	7.13349
3.64	13.2496	1.90788	6.03324	48.2285	1.53827	3.31411	7.14004
3.65	13.3225	1.91050	6.04152	48.6271	1.53968	3.31714	7.14657
3.66	13.3956	1.91311	6.04979	49.0279	1.54109	3.32017	7.15309
3.67	13.4689	1.91572	6.05805	49.4309	1.54249	3.32319	7.15960
3.68	13.5424	1.91833	6.06630	49.8360	1.54389	3.32621	7.16610
3.69	13.6161	1.92094	6.07454	50.2434	1.54529	3.32922	7.17258
3.70	13.6900	1.92354	6.08276	50.6530	1.54668	3.33222	7.17905
3.71	13.7641	1.92614	6.09098	51.0648	1.54807	3.33522	7.18552
3.72	13.8384	1.92873	6.09918	51.4788	1.54946	3.33822	7.19197
3.73	13.9129	1.93132	6.10737	51.8951	1.55085	3.34120	7.19840
3.74	13.9876	1.93391	6.11555	52.3136	1.55223	3.34419	7.20483
3.75	14.0625	1.93649	6.12372	52.7344	1.55362	3.34716	7.21125
3.76	14.1376	1.93907	6.13188	53.1574	1.55500	3.35014	7.21765
3.77	14.2129	1.94165	6.14003	53.5826	1.55637	3.35310	7.22405
3.78	14.2884	1.94422	6.14817	54.0102	1.55775	3.35607	7.23043
3.79	14.3641	1.94679	6.15630	54.4399	1.55912	3.35902	7.23680
3.80	14.4400	1.94936	6.16441	54.8720	1.56049	3.36198	7.24316
3.81	14.5161	1.95192	6.17252	55.3063	1.56186	3.36492	7.24950
3.82	14.5924	1.95448	6.18061	55.7430	1.56322	3.36786	7.25584
3.83	14.6689	1.95704	6.18870	56.1819	1.56459	3.37080	7.26217
3.84	14.7456	1.95959	6.19677	56.6231	1.56595	3.37373	7.26848
3.85	14.8225	1.96214	6.20484	57.0666	1.56731	3.37666	7.27479
3.86	14.8996	1.96469	6.21289	57.5125	1.56866	3.37958	7.28108
3.87	14.9769	1.96723	6.22093	57.9606	1.57001	3.38249	7.28736
3.88	15.0544	1.96977	6.22896	58.4111	1.57137	3.38540	7.29363
3.89	15.1321	1.97231	6.23699	58.8639	1.57271	3.38831	7.29989
3.90	15.2100	1.97484	6.24500	59.3190	1.57406	3.39121	7.30614
3.91	15.2881	1.97737	6.25300	59.7765	1.57541	3.39411	7.31238
3.92	15.3664	1.97990	6.26099	60.2363	1.57675	3.39700	7.31861
3.93	15.4449	1.98242	6.26897	60.6985	1.57809	3.39988	7.32483
3.94	15.5236	1.98494	6.27694	61.1630	1.57942	3.40277	7.33104
3.95	15.6025	1.98746	6.28490	61.6299	1.58076	3.40564	7.33723
3.96	15.6816	1.98997	6.29285	62.0991	1.58209	3.40851	7.34342
3.97	15.7609	1.99249	6.30079	62.5708	1.58342	3.41138	7.34960
3.98	15.8404	1.99499	6.30872	63.0448	1.58475	3.41424	7.35576
3.99	15.9201	1.99750	6.31664	63.5212	1.58608	3.41710	7.36192

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
4.00	16.0000	2.00000	6.32456	64.0000	1.58740	3.41995	7.36806
4.01	16.0801	2.00250	6.33246	64.4812	1.58872	3.42280	7.37420
4.02	16.1604	2.00499	6.34035	64.9648	1.59004	3.42564	7.38032
4.03	16.2409	2.00749	6.34823	65.4508	1.59136	3.42848	7.38644
4.04	16.3216	2.00998	6.35610	65.9393	1.59267	3.43131	7.39254
4.05	16.4025	2.01246	6.36396	66.4301	1.59399	3.43414	7.39864
4.06	16.4836	2.01494	6.37181	66.9234	1.59530	3.43697	7.40472
4.07	16.5649	2.01742	6.37966	67.4191	1.59661	3.43979	7.41080
4.08	16.6464	2.01990	6.38749	67.9173	1.59791	3.44260	7.41686
4.09	16.7281	2.02237	6.39531	68.4179	1.59922	3.44541	7.42291
4.10	16.8100	2.02485	6.40312	68.9210	1.60052	3.44822	7.42896
4.11	16.8921	2.02731	6.41093	69.4265	1.60182	3.45102	7.43499
4.12	16.9744	2.02978	6.41872	69.9345	1.60312	3.45382	7.44102
4.13	17.0569	2.03224	6.42651	70.4450	1.60441	3.45661	7.44703
4.14	17.1396	2.03470	6.43428	70.9579	1.60571	3.45939	7.45304
4.15	17.2225	2.03715	6.44205	71.4734	1.60700	3.46218	7.45904
4.16	17.3056	2.03961	6.44981	71.9913	1.60829	3.46496	7.46502
4.17	17.3889	2.04206	6.45755	72.5117	1.60958	3.46773	7.47100
4.18	17.4724	2.04450	6.46529	73.0346	1.61086	3.47050	7.47697
4.19	17.5561	2.04695	6.47302	73.5601	1.61215	3.47327	7.48292
4.20	17.6400	2.04939	6.48074	74.0880	1.61343	3.47603	7.48887
4.21	17.7241	2.05183	6.48845	74.6185	1.61471	3.47878	7.49481
4.22	17.8084	2.05426	6.49615	75.1514	1.61599	3.48154	7.50074
4.23	17.8929	2.05670	6.50384	75.6870	1.61726	3.48428	7.50666
4.24	17.9776	2.05913	6.51153	76.2250	1.61853	3.48703	7.51257
4.25	18.0625	2.06155	6.51920	76.7656	1.61981	3.48977	7.51847
4.26	18.1476	2.06398	6.52687	77.3088	1.62108	3.49250	7.52437
4.27	18.2329	2.06640	6.53452	77.8545	1.62234	3.49523	7.53025
4.28	18.3184	2.06882	6.54217	78.4028	1.62361	3.49796	7.53612
4.29	18.4041	2.07123	6.54981	78.9536	1.62487	3.50068	7.54199
4.30	18.4900	2.07364	6.55744	79.5070	1.62613	3.50340	7.54784
4.31	18.5761	2.07605	6.56506	80.0630	1.62739	3.50611	7.55369
4.32	18.6624	2.07846	6.57267	80.6216	1.62865	3.50882	7.55953
4.33	18.7489	2.08087	6.58027	81.1827	1.62991	3.51153	7.56535
4.34	18.8356	2.08327	6.58787	81.7465	1.63116	3.51423	7.57117
4.35	18.9225	2.08567	6.59545	82.3129	1.63241	3.51692	7.57698
4.36	19.0096	2.08806	6.60303	82.8819	1.63366	3.51962	7.58279
4.37	19.0969	2.09045	6.61060	83.4535	1.63491	3.52231	7.58858
4.38	19.1844	2.09284	6.61816	84.0277	1.63619	3.52499	7.59436
4.39	19.2721	2.09523	6.62571	84.6045	1.63740	3.52767	7.60014
4.40	19.3600	2.09762	6.63325	85.1840	1.63864	3.53035	7.60590
4.41	19.4481	2.10000	6.64078	85.7661	1.63988	3.53302	7.61166
4.42	19.5364	2.10238	6.64831	86.3509	1.64112	3.53569	7.61741
4.43	19.6249	2.10476	6.65582	86.9383	1.64236	3.53835	7.62315
4.44	19.7136	2.10713	6.66333	87.5284	1.64359	3.54101	7.62888
4.45	19.8025	2.10950	6.67083	88.1211	1.64483	3.54367	7.63461
4.46	19.8916	2.11187	6.67832	88.7165	1.64606	3.54632	7.64032
4.47	19.9809	2.11424	6.68581	89.3146	1.64729	3.54897	7.64603
4.48	20.0704	2.11660	6.69328	89.9154	1.64851	3.55162	7.65172
4.49	20.1601	2.11896	6.70075	90.5188	1.64974	3.55426	7.65741

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
4.50	20.2500	2.12132	6.70820	91.1250	1.65096	3.55689	7.66309
4.51	20.3401	2.12368	6.71565	91.7339	1.65219	3.55953	7.66877
4.52	20.4304	2.12603	6.72309	92.3454	1.65341	3.56215	7.67443
4.53	20.5209	2.12838	6.73053	92.9597	1.65462	3.56478	7.68009
4.54	20.6116	2.13073	6.73795	93.5767	1.65584	3.56740	7.68573
4.55	20.7025	2.13307	6.74537	94.1964	1.65706	3.57002	7.69137
4.56	20.7936	2.13542	6.75278	94.8188	1.65827	3.57263	7.69700
4.57	20.8849	2.13776	6.76018	95.4440	1.65948	3.57524	7.70262
4.58	20.9764	2.14009	6.76757	96.0719	1.66069	3.57785	7.70824
4.59	21.0681	2.14243	6.77495	96.7026	1.66190	3.58045	7.71384
4.60	21.1600	2.14476	6.78233	97.3360	1.66310	3.58305	7.71944
4.61	21.2521	2.14709	6.78970	97.9722	1.66431	3.58564	7.72503
4.62	21.3444	2.14942	6.79706	98.6111	1.66551	3.58823	7.73061
4.63	21.4369	2.15174	6.80441	99.2528	1.66671	3.59082	7.73619
4.64	21.5296	2.15407	6.81175	99.8973	1.66791	3.59340	7.74175
4.65	21.6225	2.15639	6.81909	100.545	1.66911	3.59598	7.74731
4.66	21.7156	2.15870	6.82642	101.195	1.67030	3.59856	7.75286
4.67	21.8089	2.16102	6.83374	101.848	1.67150	3.60113	7.75840
4.68	21.9024	2.16333	6.84105	102.503	1.67269	3.60370	7.76394
4.69	21.9961	2.16564	6.84836	103.162	1.67388	3.60626	7.76946
4.70	22.0900	2.16795	6.85565	103.823	1.67507	3.60883	7.77498
4.71	22.1841	2.17025	6.86294	104.487	1.67626	3.61138	7.78049
4.72	22.2784	2.17256	6.87023	105.154	1.67744	3.61394	7.78599
4.73	22.3729	2.17486	6.87750	105.824	1.67863	3.61649	7.79149
4.74	22.4676	2.17715	6.88477	106.496	1.67981	3.61903	7.79697
4.75	22.5625	2.17945	6.89202	107.172	1.68099	3.62158	7.80245
4.76	22.6576	2.18174	6.89928	107.850	1.68217	3.62412	7.80793
4.77	22.7529	2.18403	6.90652	108.531	1.68334	3.62665	7.81339
4.78	22.8484	2.18632	6.91375	109.215	1.68452	3.62919	7.81885
4.79	22.9441	2.18861	6.92098	109.902	1.68569	3.63172	7.82429
4.80	23.0400	2.19089	6.92820	110.592	1.68687	3.63424	7.82974
4.81	23.1361	2.19317	6.93542	111.285	1.68804	3.63676	7.83517
4.82	23.2324	2.19545	6.94262	111.980	1.68920	3.63928	7.84059
4.83	23.3289	2.19773	6.94982	112.679	1.69037	3.64180	7.84601
4.84	23.4256	2.20000	6.95701	113.380	1.69154	3.64431	7.85142
4.85	23.5225	2.20227	6.96419	114.084	1.69270	3.64682	7.85683
4.86	23.6196	2.20454	6.97137	114.791	1.69386	3.64932	7.86222
4.87	23.7169	2.20681	6.97854	115.501	1.69503	3.65182	7.86761
4.88	23.8144	2.20907	6.98570	116.214	1.69619	3.65432	7.87299
4.89	23.9121	2.21133	6.99285	116.930	1.69734	3.65681	7.87837
4.90	24.0100	2.21359	7.00000	117.649	1.69850	3.65931	7.88374
4.91	24.1081	2.21585	7.00714	118.371	1.69965	3.66179	7.88909
4.92	24.2064	2.21811	7.01427	119.095	1.70081	3.66428	7.89445
4.93	24.3049	2.22036	7.02140	119.823	1.70196	3.66676	7.89979
4.94	24.4036	2.22261	7.02851	120.554	1.70311	3.66924	7.90513
4.95	24.5025	2.22486	7.03562	121.287	1.70426	3.67171	7.91046
4.96	24.6016	2.22711	7.04273	122.024	1.70540	3.67418	7.91578
4.97	24.7009	2.22935	7.04982	122.763	1.70655	3.67665	7.92110
4.98	24.8004	2.23159	7.05691	123.506	1.70769	3.67911	7.92641
4.99	24.9001	2.23383	7.06399	124.251	1.70884	3.68157	7.93171

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
5.00	25.0000	2.23607	7.07107	125.000	1.70998	3.68403	7.93701
5.01	25.1001	2.23830	7.07814	125.752	1.71112	3.68649	7.94229
5.02	25.2004	2.24054	7.08520	126.506	1.71225	3.68894	7.94757
5.03	25.3009	2.24277	7.09225	127.264	1.71339	3.69138	7.95285
5.04	25.4016	2.24499	7.09930	128.024	1.71452	3.69383	7.95811
5.05	25.5025	2.24722	7.10634	128.788	1.71566	3.69627	7.96337
5.06	25.6036	2.24944	7.11337	129.554	1.71679	3.69871	7.96863
5.07	25.7049	2.25167	7.12039	130.324	1.71792	3.70114	7.97387
5.08	25.8064	2.25389	7.12741	131.097	1.71905	3.70357	7.97911
5.09	25.9081	2.25610	7.13442	131.872	1.72017	3.70600	7.98434
5.10	26.0100	2.25832	7.14143	132.651	1.72130	3.70843	7.98957
5.11	26.1121	2.26053	7.14843	133.433	1.72242	3.71085	7.99479
5.12	26.2144	2.26274	7.15542	134.218	1.72355	3.71327	8.00000
5.13	26.3169	2.26495	7.16240	135.006	1.72467	3.71569	8.00520
5.14	26.4196	2.26716	7.16938	135.797	1.72579	3.71810	8.01040
5.15	26.5225	2.26936	7.17635	136.591	1.72691	3.72051	8.01559
5.16	26.6256	2.27156	7.18331	137.388	1.72802	3.72292	8.02078
5.17	26.7289	2.27376	7.19027	138.188	1.72914	3.72532	8.02596
5.18	26.8324	2.27596	7.19722	138.992	1.73025	3 72772	8.03113
5.19	26.9361	2.27816	7.20417	139.798	1.73137	3.73012	8.03629
5.20	27.0400	2.28035	7.21110	140.608	1.73248	3.73251	8.04145
5.21	27.1441	2.28254	7.21803	141.421	1.73359	3.73490	8.04660
5.22	27.2484	2.28473	7.22496	142.237	1.73470	3.73729	8.05175
5.23	27.3529	2.28692	7.23187	143.056	1.73580	3.73968	8.05689
5.24	27.4576	2.28910	7.23878	143.878	1.73691	3.74206	8.06202
5.25	27.5625	2.29129	7.24569	144.703	1 73801	3.74443	8.06714
5.26	27.6676	2.29347	7.25259	145.532	1.73912	3.74681	8.07226
5.27	27.7729	2.29565	7.25948	146.363	1.74022	3.74918	8.07737
5.28	27.8784	2.29783	7.26636	147.198	1.74132	3.75155	8.08248
5.29	27.9841	2.30000	7.27324	148.036	1.74242	3.75392	8.08758
5.30	28.0900	2.30217	7.28011	148.877	1.74351	3.75629	8.09267
5.31	28.1961	2.30434	7.28697	149.721	1.74461	3.75865	8.09776
5.32	28.3024	2.30651	7.29383	150.569	1.74570	3.76101	8.10284
5.33	28.4089	2.30868	7.30068	151.419	1.74680	3.76336	8.10791
5.34	28.5156	2.31084	7.30753	152.273	1.74789	3.76571	8.11298
5.35	28.6225	2.31301	7.31437	153.130	1.74898	3.76806	8.11804
5.36	28.7296	2.31517	7.32120	153.991	1.75007	3.77041	8.12310
5.37	28.8369	2.31733	7.32803	154.854	1.75116	3.77275	8.12814
5.38	28.9444	2.31948	7.33485	155.721	1.75224	3.77509	8.13319
5.39	29.0521	2.32164	7.34166	156.591	1.75333	3.77743	8.13822
5.40	29.1600	2.32379	7.34847	157.464	1.75441	3.77976	8.14325
5.41	29.2681	2.32594	7.35527	158.340	1.75549	3.78209	8.14828
5.42	29.3764	2.32809	7.36206	159.220	1.75657	3.78442	8.15329
5.43	29.4849	2.33024	7.36885	160.103	1.75765	3.78675	8.15831
5.44	29.5936	2.33238	7.37564	160.989	1.75873	3.78907	8.16331
5.45	29.7025	2.33452	7.38241	161.879	1.75981	3.79139	8.16831
5.46	29.8116	2.33666	7.38918	162.771	1.76088	3.79371	8.17330
5.47	29.9209	2.33880	7.39594	163.667	1.76196	3.79603	8.17829
5.48	30.0304	2.34094	7.40270	164.567	1.76303	3.79834	8.18327
5.49	30.1401	2.34307	7.40945	165.469	1.76410	3.80065	8.18824

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
5.50	30.2500	2.34521	7.41620	166.375	1.76517	3.80295	8.19321
5.51	30.3601	2.34734	7.42294	167.284	1.76624	3.80526	8.19818
5.52	30.4704	2.34947	7.42967	168.197	1.76731	3.80756	8.20313
5.53	30.5809	2.35160	7.43640	169.112	1.76838	3.80985	8.20808
5.54	30.6916	2.35372	7.44312	170.031	1.76944	3.81215	8.21303
5.55	30.8025	2.35584	7.44983	170.954	1.77051	3.81444	8.21797
5.56	30.9136	2.35797	7.45654	171.880	1.77157	3.81673	8.22290
5.57	31.0249	2.36008	7.46324	172.809	1.77263	3.81902	8.22783
5.58	31.1364	2.36220	7.46994	173.741	1.77369	3.82130	8.23275
5.59	31.2481	2.36432	7.47663	174.677	1.77475	3.82358	8.23766
5.60	31.3600	2.36643	7.48331	175.616	1.77581	3.82586	8.24257
5.61	31.4721	2.36854	7.48999	176.558	1.77686	3.82814	8.24747
5.62	31.5844	2.37065	7.49667	177.504	1.77792	3.83041	8.25237
5.63	31.6969	2.37276	7.50333	178.454	1.77897	3.83268	8.25726
5.64	31.8096	2.37487	7.50999	179.406	1.78003	3.83495	8.26215
5.65	31.9225	2.37697	7.51665	180.362	1.78108	3.83722	8.26703
5.66	32.0356	2.37908	7.52330	181.321	1.78213	3.83948	8.27190
5.67	32.1489	2.38118	7.52994	182.284	1.78318	3.84174	8.27677
5.68	32.2624	2.38328	7.53658	183.250	1.78422	3.84399	8.28164
5.69	32.3761	2.38537	7.54321	184.220	1.78527	3.84625	8.28649
5.70	32.4900	2.38747	7.54983	185.193	1.78632	3.84850	8.29134
5.71	32.6041	2.38956	7.55645	186.169	1.78736	3.85075	8.29619
5.72	32.7184	2.39165	7.56307	187.149	1.78840	3.85300	8.30103
5.73	32.8329	2.39374	7.56968	188.133	1.78944	3.85524	8.30587
5.74	32.9476	2.39583	7.57628	189.119	1.79048	3.85748	8.31069
5.75	33.0625	2.39792	7.58288	190.109	1.79152	3.85972	8.31552
5.76	33.1776	2.40000	7.58947	191.103	1.79256	3.86196	8.32034
5.77	33.2929	2.40208	7.59605	192.100	1.79360	3.86419	8.32515
5.78	33.4084	2.40416	7.60263	193.101	1.79463	3.86642	8.32995
5.79	33.5241	2.40624	7.60920	194.105	1.79567	3.86865	8.33476
5.80	33.6400	2.40832	7.61577	195.112	1.79670	3.87088	8.33955
5.81	33.7561	2.41039	7.62234	196.123	1.79773	3.87310	8.34434
5.82	33.8724	2.41247	7.62889	197.137	1.79876	3.87532	8.34913
5.83	33.9889	2.41454	7.63544	198.155	1.79979	3.87754	8.35390
5.84	34.1056	2.41661	7.64199	199.177	1.80082	3.87975	8.35868
5.85	34.2225	2.41868	7.64853	200.202	1.80185	3.88197	8.36345
5.86	34.3396	2.42074	7.65506	201.230	1.80288	3.88418	8.36821
5.87	34.4569	2.42281	7.66159	202.262	1.80390	3.88639	8.37297
5.88	34.5744	2.42487	7.66812	203.297	1.80492	3.88859	8.37772
5.89	34.6921	2.42693	7.67463	204.336	1.80595	3.89080	8.38247
5.90	34.8100	2.42899	7.68115	205.379	1.80697	3.89300	8.38721
5.91	34.9281	2.43105	7.68765	206.425	1.80799	3.89519	8.39194
5.92	35.0464	2.43311	7.69415	207.475	1.80901	3.89739	8.39667
5.93	35.1649	2.43516	7.70065	208.528	1.81003	3.89958	8.40140
5.94	35.2836	2.43721	7.70714	209.585	1.81104	3.90177	8.40612
5.95	35.4025	2.43926	7.71362	210.645	1.81206	3.90396	8.41083
5.96	35.5216	2.44131	7.72010	211.709	1.81307	3.90615	8.41554
5.97	35.6409	2.44336	7.72658	212.776	1.81409	3.90833	8.42025
5.98	35.7604	2.44540	7.73305	213.847	1.81510	3.91051	8.42494
5.99	35.8801	2.44745	7.73951	214.922	1.81611	3.91269	8.42964

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
6.00	36.0000	2.44949	7.74597	216.000	1.81712	3.91487	8.43433
6.01	36.1201	2.45153	7.75242	217.082	1.81813	3.91704	8.43901
6.02	36.2404	2.45357	7.75887	218.167	1.81914	3.91921	8.44369
6.03	36.3609	2.45561	7.76531	219.256	1.82014	3.92138	8.44836
6.04	36.4816	2.45764	7.77174	220.349	1.82115	3.92355	8.45303
6.05	36.6025	2.45967	7.77817	221.445	1.82215	3.92571	8.45769
6.06	36.7236	2.46171	7.78460	222.545	1.82316	3.92787	8.46235
6.07	36.8449	2.46374	7.79102	223.649	1.82416	3.93003	8.46700
6.08	36.9664	2.46577	7.79744	224.756	1.82516	3.93219	8.47165
6.09	37.0881	2.46779	7.80385	225.867	1.82616	3.93434	8.47629
6.10	37.2100	2.46982	7.81025	226.981	1.82716	3.93650	8.48093
6.11	37.3321	2.47184	7.81665	228.099	1.82816	3.93865	8.48556
6.12	37.4544	2.47386	7.82304	229.221	1.82915	3.94079	8.49018
6.13	37.5769	2.47588	7.82943	230.346	1.83015	3.94294	8.49481
6.14	37.6996	2.47790	7.83582	231.476	1.83115	3.94508	8.49942
6.15	37.8225	2.47992	7.84219	232.608	1.83214	3.94722	8.50403
6.16	37.9456	2.48193	7.84857	233.745	1.83313	3.94936	8.50864
6.17	38.0689	2.48395	7.85493	234.885	1.83412	3.95150	8.51324
6.18	38.1924	2.48596	7.86130	236.029	1.83511	3.95363	8.51784
6.19	38.3161	2.48797	7.86766	237.177	1.83610	3.95576	8.52243
6.20	38.4400	2.48998	7.87401	238.328	1.83709	3.95789	8.52702
6.21	38.5641	2.49199	7.88036	239.483	1.83808	3.96002	8.53160
6.22	38.6884	2.49399	7.88670	240.642	1.83906	3.96214	8.53618
6.23	38.8129	2.49600	7.89303	241.804	1.84005	3.96427	8.54075
6.24	38.9376	2.49800	7.89937	242.971	1.84103	3.96638	8.54532
6.25	39.0625	2.50000	7.90569	244.141	1.84202	3.96850	8.54988
6.26	39.1876	2.50200	7.91202	245.314	1.84300	3.97062	8.55444
6.27	39.3129	2.50400	7.91833	246.492	1.84398	3.97273	8:55899
6.28	39.4384	2.50599	7.92465	247.673	1.84496	3.97484	8.56354
6.29	39.5641	2.50799	7.93095	248.858	1.84594	3.97695	8.56808
6.30	39.6900	2.50998	7.93725	250.047	1.84691	3.97906	8.57262
6.31	39.8161	2.51197	7.94355	251.240	1.84789	3.98116	8.57715
6.32	39.9424	2.51396	7.94984	252.436	1.84887	3.98326	8.58168
6.33	40.0689	2.51595	7.95613	253.636	1.84984	3.98536	8.58620
6.34	40.1956	2.51794	7.96241	254.840	1.85082	3.98746	8.59072
6.35	40.3225	2.51992	7.96869	256.048	1.85179	3.98956	8.59524
6.36	40.4496	2.52190	7.97496	257.259	1.85276	3.99165	8.59975
6.37	40.5769	2.52389	7.98123	258.475	1.85373	3.99374	8.60425
6.38	40.7044	2.52587	7.98749	259.694	1.85470	3.99583	8.60875
6.39	40.8321	2.52784	7.99375	260.917	1.85567	3.99792	8.61325
6.40	40.9600	2.52982	8.00000	262.144	1.85664	4.00000	8.61774
6.41	41.0881	2.53180	8.00625	263.375	1.85760	4.00208	8.62222
6.42	41.2164	2.53377	8.01249	264.609	1.85857	4.00416	8.62671
6.43	41.3449	2.53574	8.01873	265.848	1.85953	4.00624	8.63118
6.44	41.4736	2.53772	8.02496	267.090	1.86050	4.00832	8.63566
6.45	41.6025	2.53969	8.03119	268.336	1.86146	4.01039	8.64012
6.46	41.7316	2.54165	8.03741	269.586	1.86242	4.01246	8.64459
6.47	41.8609	2.54362	8.04363	270.840	1.86338	4.01453	8.64904
6.48	41.9904	2.54558	8.04984	272.098	1.86434	4.01660	8.65350
6.49	42.1201	2.54755	8.05605	273.359	1.86530	4.01866	8.65795

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
6.50	42.2500	2.54951	8.06226	274.625	1.86626	4.02073	8.66239
6.51	42.3801	2.55147	8.06846	275.894	1.86721	4.02279	8.66683
6.52	42.5104	2.55343	8.07465	277.168	1.86817	4.02485	8.67127
6.53	42.6409	2.55539	8.08084	278.445	1.86912	4.02690	8.67570
6.54	42.7716	2.55734	8.08703	279.726	1.87008	4.02896	8.68012
6.55	42.9025	2.55930	8.09321	281.011	1.87103	4.03101	8.68455
6.56	43.0336	2.56125	8.09938	282.300	1.87198	4.03306	8.68896
6.57	43.1649	2.56320	8.10555	283.593	1.87293	4.03511	8.69338
6.58	43.2964	2.56515	8.11172	284.890	1.87388	4.03715	8.69778
6.59	43.4281	2.56710	8.11788	286.191	1.87483	4.03920	8.70219
6.60	43.5600	2.56905	8.12404	287.496	1.87578	4.04124	8.70659
6.61	43.6921	2.57099	8.13019	288.805	1.87672	4.04328	8.71098
6.62	43.8244	2.57294	8.13634	290.118	1.87767	4.04532	8.71537
6.63	43.9569	2.57488	8.14248	291.434	1.87862	4.04735	8.71976
6.64	44.0896	2.57682	8.14862	292.755	1.87956	4.04939	8.72414
6.65	44.2225	2.57876	8.15475	294.080	1.88050	4.05142	8.72852
6.66	44.3556	2.58070	8.16088	295.408	1.88144	4.05345	8.73289
6.67	44.4889	2.58263	8.16701	296.741	1.88239	4.05548	8.73726
6.68	44.6224	2.58457	8.17313	298.078	1.88333	4.05750	8.74162
6.69	44.7561	2.58650	8.17924	299.418	1.88427	4.05953	8.74598
6.70	44.8900	2.58844	8.18535	300.763	1.88520	4.06155	8.75034
6.71	45.0241	2.59037	8.19146	302.112	1.88614	4.06357	8.75469
6.72	45.1584	2.59230	8.19756	303.464	1.88708	4.06559	8.75904
6.73	45.2929	2.59422	8.20366	304.821	1.88801	4.06760	8.76338
6.74	45.4276	2.59615	8.20975	306.182	1.88895	4.06961	8.76772
6.75	45.5625	2.59808	8.21584	307.547	1.88988	4.07163	8.77205
6.76	45.6976	2.60000	8.22192	308.916	1.89081	4.07364	8.77638
6.77	45.8329	2.60192	8.22800	310.289	1.89175	4.07564	8.78071
6.78	45.9684	2.60384	8.23408	311.666	1.89268	4.07765	8.78503
6.79	46.1041	2.60576	8.24015	313.047	1.89361	4.07965	8.78935
6.80	46.2400	2.60768	8.24621	314.432	1.89454	4.08166	8.79366
6.81	46.3761	2.60960	8.25227	315.821	1.89546	4.08365	8.79797
6.82	46.5124	2.61151	8.25833	317.215	1.89639	4.08565	8.80227
6.83	46.6489	2.61343	8.26438	318.612	1.89732	4.08765	8.80657
6.84	46.7856	2.61534	8.27043	320.014	1.89824	4.08964	8.81087
6.85	46.9225	2.61725	8.27647	321.419	1.89917	4.09163	8.81516
6.86	47.0596	2.61916	8.28251	322.829	1.90009	4.09362	8.81945
6.87	47.1969	2.62107	8.28855	324.243	1.90102	4.09561	8.82373
6.88	47.3344	2.62298	8.29458	325.661	1.90194	4.09760	8.82801
6.89	47.4721	2.62488	8.30060	327.083	1.90286	4.09958	8.83228
6.90	47.6100	2.62679	8.30662	328.509	1.90378	4.10157	8.83656
6.91	47.7481	2.62869	8.31264	329.939	1.90470	4.10355	8.84082
6.92	47.8864	2.63059	8.31865	331.374	1.90562	4.10552	8.84509
6.93	48.0249	2.63249	8.32466	332.813	1.90653	4.10750	8.84934
6.94	48.1636	2.63439	8.33067	334.255	1.90745	4.10948	8.85360
6.95	48.3025	2.63629	8.33667	335.702	1.90837	4.11145	8.85785
6.96	48.4416	2.63818	8.34266	337.154	1.90928	4.11342	8.86210
6.97	48.5809	2.64008	8.34865	338.609	1.91019	4.11539	8.86634
6.98	48.7204	2.64197	8.35464	340.068	1.91111	4.11736	8.87058
6.99	48.8601	2.64386	8.36062	341.532	1.91202	4.11932	8.87481

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
7.00	49.0000	2.64575	8.36660	343.000	1.91293	4.12129	8.87904
7.01	49.1401	2.64764	8.37257	344.472	1.91384	4.12325	8.88327
7.02	49.2804	2.64953	8.37854	345.948	1.91475	4.12521	8.88749
7.03	49.4209	2.65141	8.38451	347.429	1.91566	4.12716	8.89171
7.04	49.5616	2.65330	8.39047	348.914	1.91657	4.12912	8.89592
7.05	49.7025	2.65518	8.39643	350.403	1.91747	4.13107	8.90013
7.06	49.8436	2.65707	8.40238	351.896	1.91838	4.13303	8.90434
7.07	49.9849	2.65895	8.40833	353.393	1.91929	4.13498	8.90854
7.08	50.1264	2.66083	8.41427	354.895	1.92019	4.13693	8.91274
7.09	50.2681	2.66271	8.42021	356.401	1.92109	4.13887	8.91693
7.10	50.4100	2.66458	8.42615	357.911	1.92200	4.14082	8.92112
7.11	50.5521	2.66646	8.43208	359.425	1.92290	4.14276	8.92531
7.12	50.6944	2.66833	8.43801	360.944	1.92380	4.14470	8.92949
7.13	50.8369	2.67021	8.44393	362.467	1.92470	4.14664	8.93367
7.14	50.9796	2.67208	8.44985	363.994	1.92560	4.14858	8.93784
7.15	51.1225	2.67395	8.45577	365.526	1.92650	4.15052	8.94201
7.16	51.2656	2.67582	8.46168	367.062	1.92740	4.15245	8.94618
7.17	51.4089	2.67769	8.46759	368.602	1.92829	4.15438	8.95034
7.18	51.5524	2.67955	8.47349	370.146	1.92919	4.15631	8.95450
7.19	51.6961	2.68142	8.47939	371.695	1.93008	4.15824	8.95866
7.20	51.8400	2.68328	8.48528	373.248	1.93098	4.16017	8.96281
7.21	51.9841	2.68514	8.49117	374.805	1.93187	4.16209	8.96696
7.22	52.1284	2.68701	8.49706	376.367	1.93277	4.16402	8.97110
7.23	52.2729	2.68887	8.50294	377.933	1.93366	4.16594	8.97524
7.24	52.4176	2.69072	8.50882	379.503	1.93455	4.16786	8.97938
7.25	52.5625	2.69258	8.51469	381.078	1.93544	4.16978	8.98351
7.26	52.7076	2.69444	8.52056	382.657	1.93633	4.17169	8.98764
7.27	52.8529	2.69629	8.52643	384.241	1.93722	4.17361	8.99176
7.28	52.9984	2.69815	8.53229	385.828	1.93810	4.17552	8.99588
7.29	53.1441	2.70000	8.53815	387.420	1.93899	4.17743	9.00000
7.30	53.2900	2.70185	8.54400	389.017	1.93988	4.17934	9.00411
7.31	53.4361	2.70370	8.54985	390.618	1.94076	4.18125	9.00822
7.32	53.5824	2.70555	8.55570	392.223	1.94165	4.18315	9.01233
7.33	53.7289	2.70740	8.56154	393.833	1.94253	4.18506	9.01643
7.34	53.8756	2.70924	8.56738	395.447	1.94341	4.18696	9.02053
7.35	54.0225	2.71109	8.57321	397.065	1.94430	4.18886	9.02462
7.36	54.1696	2.71293	8.57904	398.688	1.94518	4.19076	9.02871
7.37	54.3169	2.71477	8.58487	400.316	1.94606	4.19266	9.03280
7.38	54.4644	2.71662	8.59069	401.947	1.94694	4.19455	9.03689
7.39	54.6121	2.71846	8.59651	403.583	1.94782	4.19644	9.04097
7.40	54.7600	2.72029	8.60233	405.224	1.94870	4.19834	9.04504
7.41	54.9081	2.72213	8.60814	406.869	1.94957	4.20023	9.04911
7.42	55.0564	2.72397	8.61394	408.518	1.95045	4.20212	9.05318
7.43	55.2049	2.72580	8.61974	410.172	1.95132	4.20400	9.05725
7.44	55.3536	2.72764	8.62554	411.831	1.95220	4.20589	9.06131
7.45	55.5025	2.72947	8.63134	413.494	1.95307	4.20777	9.06537
7.46	55.6516	2.73130	8.63713	415.161	1.95395	4.20965	9.06942
7.47	55.8009	2.73313	8.64292	416.833	1.95482	4.21153	9.07347
7.48	55.9504	2.73496	8.64870	418.509	1.95569	4.21341	9.07752
7.49	56.1001	2.73679	8.65448	420.190	1.95656	4.21529	9.08156

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
7.50	56.2500	2.73861	8.66025	421.875	1.95743	4.21716	9.08560
7.51	56.4001	2.74044	8.66603	423.565	1.95830	4.21904	9.08964
7.52	56.5504	2.74226	8.67179	425.259	1.95917	4.22091	9.09367
7.53	56.7009	2.74408	8.67756	426.958	1.96004	4.22278	9.09770
7.54	56.8516	2.74591	8.68332	428.661	1.96091	4.22465	9.10173
7.55	57.0025	2.74773	8.68907	430.369	1.96177	4.22651	9.10575
7.56	57.1536	2.74955	8.69483	432.081	1.96264	4.22838	9.10977
7.57	57.3049	2.75136	8.70057	433.798	1.96350	4.23024	9.11378
7.58	57.4564	2.75318	8.70632	435.520	1.96437	4.23210	9.11779
7.59	57.6081	2.75500	8.71206	437.245	1.96523	4.23396	9.12180
7.60	57.7600	2.75681	8.71780	438.976	1.96610	4.23582	9.12581
7.61	57.9121	2.75862	8.72353	440.711	1.96696	4.23768	9.12981
7.62	58.0644	2.76043	8.72926	442.451	1.96782	4.23954	9.13380
7.63	58.2169	2.76225	8.73499	444.195	1.96868	4.24139	9.13780
7.64	58.3696	2.76405	8.74071	445.944	1.96954	4.24324	9.14179
7.65	58.5225	2.76586	8.74643	447.697	1.97040	4.24509	9.14577
7.66	58.6756	2.76767	8.75214	449.455	1.97126	4.24694	9.14976
7.67	58.8289	2.76948	8.75785	451.218	1.97211	4.24879	9.15374
7.68	58.9824	2.77128	8.76356	452.985	1.97297	4.25063	9.15771
7.69	59.1361	2.77308	8.76926	454.757	1.97383	4.25248	9.16169
7.70	59.2900	2.77489	8.77496	456.533	1.97468	4.25432	9.16566
7.71	59.4441	2.77669	8.78066	458.314	1.97554	4.25616	9.16962
7.72	59.5984	2.77849	8.78635	460.100	1.97639	4.25800	9.17359
7.73	59.7529	2.78029	8.79204	461.890	1.97724	4.25984	9.17754
7.74	59.9076	2.78209	8.79773	463.685	1.97809	4.26167	9.18150
7.75	60.0625	2.78388	8.80341	465.484	1.97895	4.26351	9.18545
7.76	60.2176	2.78568	8.80909	467.289	1.97980	4.26534	9.18940
7.77	60.3729	2.78747	8.81476	469.097	1.98065	4.26717	9.19335
7.78	60.5284	2.78927	8.82043	470.911	1.98150	4.26900	9.19729
7.79	60.6841	2.79106	8.82610	472.729	1.98234	4.27083	9.20123
7.80	60.8400	2.79285	8.83176	474.552	1.98319	4.27266	9.20516
7.81	60.9961	2.79464	8.83742	476.380	1.98404	4.27448	9.20910
7.82	61.1524	2.79643	8.84308	478.212	1.98489	4.27631	9.21302
7.83	61.3089	2.79821	8.84873	480.049	1.98573	4.27813	9.21695
7.84	61.4656	2.80000	8.85438	481.890	1.98658	4.27995	9.22087
7.85	61.6225	2.80179	8.86002	483.737	1.98742	4.28177	9.22479
7.86	61.7796	2.80357	8.86566	485.588	1.98826	4.28359	9.22871
7.87	61.9369	2.80535	8.87130	487.443	1.98911	4.28540	9.23262
7.88	62.0944	2.80713	8.87694	489.304	1.98995	4.28722	9.23653
7.89	62.2521	2.80891	8.88257	491.169	1.99079	4.28903	9.24043
7.90	62.4100	2.81069	8.88819	493.039	1.99163	4.29084	9.24434
7.91	62.5681	2.81247	8.89382	494.914	1.99247	4.29265	9.24823
7.92	62.7264	2.81425	8.89944	496.793	1.99331	4.29446	9.25213
7.93	62.8849	2.81603	8.90505	498.677	1.99415	4.29627	9.25602
7.94	63.0436	2.81780	8.91067	500.566	1.99499	4.29807	9.25991
7.95	63.2025	2.81957	8.91628	502.460	1.99582	4.29987	9.26380
7.96	63.3616	2.82135	8.92188	504.358	1.99666	4.30168	9.26768
7.97	63.5209	2.82312	8.92749	506.262	1.99750	4.30348	9.27156
7.98	63.6804	2.82489	8.93308	508.170	1.99833	4.30528	9.27544
7.99	63.8401	2.82666	8.93868	510.082	1.99917	4.30707	9.27931

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
8.00	64.0000	2.82843	8.94427	512.000	2.00000	4.30887	9.28318
8.01	64.1601	2.83019	8.94986	513.922	2.00083	4.31066	9.28704
8.02	64.3204	2.83196	8.95545	515.850	2.00167	4.31246	9.29091
8.03	64.4809	2.83373	8.96103	517.782	2.00250	4.31425	9.29477
8.04	64.6416	2.83549	8.96660	519.718	2.00333	4.31604	9.29862
8.05	64.8025	2.83725	8.97218	521.660	2.00416	4.31783	9.30248
8.06	64.9636	2.83901	8.97775	523.607	2.00499	4.31961	9.30633
8.07	65.1249	2.84077	8.98332	525.558	2.00582	4.32140	9.31018
8.08	65.2864	2.84253	8.98888	527.514	2.00664	4.32318	9.31402
8.09	65.4481	2.84429	8.99444	529.475	2.00747	4.32497	9.31786
8.10	65.6100	2.84605	9.00000	531.441	2.00830	4.32675	9.32170
8.11	65.7721	2.84781	9.00555	533.412	2.00912	4.32853	9.32553
8.12	65.9344	2.84956	9.01110	535.387	2.00995	4.33031	9.32936
8.13	66.0969	2.85132	9.01665	537.368	2.01078	4.33208	9.33319
8.14	66.2596	2.85307	9.02219	539.353	2.01160	4.33386	9.33702
8.15	66.4225	2.85482	9.02774	541.343	2.01242	4.33563	9.34084
8.16	66.5856	2.85657	9.03327	543.338	2.01325	4.33741	9.34466
8.17	66.7489	2.85832	9.03881	545.339	2.01407	4.33918	9.34847
8.18	66.9124	2.86007	9.04434	547.343	2.01489	4.34095	9.35229
8.19	67.0761	2.86182	9.04986	549.353	2.01571	4.34271	9.35610
8.20	67.2400	2.86356	9.05539	551.368	2.01653	4.34448	9.35990
8.21	67.4041	2.86531	9.06091	553.388	2.01735	4.34625	9.36370
8.22	67.5684	2.86705	9.06642	555.412	2.01817	4.34801	9.36751
8.23	67.7329	2.86880	9.07193	557.442	2.01899	4.34977	9.37130
8.24	67.8976	2.87054	9.07744	559.476	2.01980	4.35153	9.37510
8.25	68.0625	2.87228	9.08295	561.516	2.02062	4.35329	9.37889
8.26	68.2276	2.87402	9.08845	563.560	2.02144	4.35505	9.38268
8.27	68.3929	2.87576	9.09395	565.609	2.02225	4.35681	9.38646
8.28	68.5584	2.87750	9.09945	567.664	2.02307	4.35856	9.39024
8.29	68.7241	2.87924	9.10494	569.723	2.02388	4.36032	9.39402
8.30	68.8900	2.88097	9.11043	571.787	2.02469	4.36207	9.39780
8.31	69.0561	2.88271	9.11592	573.856	2.02551	4.36382	9.40157
8.32	69.2224	2.88444	9.12140	575.930	2.02632	4.36557	9.40534
8.33	69.3889	2.88617	9.12688	578.010	2.02713	4.36732	9.40911
8.34	69.5556	2.88791	9.13236	580.094	2.02794	4.36907	9.41287
8.35	69.7225	2.88964	9.13783	582.183	2.02875	4.37081	9.41663
8.36	69.8896	2.89137	9.14330	584.277	2.02956	4.37256	9.42039
8.37	70.0569	2.89310	9.14877	586.376	2.03037	4.37430	9.42414
8.38	70.2244	2.89482	9.15423	588.480	2.03118	4.37604	9.42789
8.39	70.3921	2.89655	9.15969	590.590	2.03199	4.37778	9.43164
8.40	70.5600	2.89828	9.16515	592.704	2.03279	4.37952	9.43539
8.41	70.7281	2.90000	9.17061	594.823	2.03360	4.38126	9.43913
8.42	70.8964	2.90172	9.17606	596.948	2.03440	4.38299	9.44287
8.43	71.0649	2.90345	9.18150	599.077	2.03521	4.38473	9.44661
8.44	71.2336	2.90517	9.18695	601.212	2.03601	4.38646	9.45034
8.45	71.4025	2.90689	9.19239	603.351	2.03682	4.38819	9.45407
8.46	71.5716	2.90861	9.19783	605.496	2.03762	4.38992	9.45780
8.47	71.7409	2.91033	9.20326	607.645	2.03842	4.39165	9.46152
8.48	71.9104	2.91204	9.20869	609.800	2.03923	4.39338	9.46525
8.49	72.0801	2.91376	9.21412	611.960	2.04003	4.39510	9.46897

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
8.50	72.2500	2.91548	9.21954	614.125	2.04083	4.39683	9.47268
8.51	72.4201	2.91719	9.22497	616.295	2.04163	4.39855	9.47640
8.52	72.5904	2.91890	9.23038	618.470	2.04243	4.40028	9.48011
8.53	72.7609	2.92062	9.23580	620.650	2.04323	4.40200	9.48381
8.54	72.9316	2.92233	9.24121	622.836	2.04402	4.40372	9.48752
8.55	73.1025	2.92404	9.24662	625.026	2.04482	4.40543	9.49122
8.56	73.2736	2.92575	9.25203	627.222	2.04562	4.40715	9.49492
8.57	73.4449	2.92746	9.25743	629.423	2.04641	4.40887	9.49861
8.58	73.6164	2.92916	9.26283	631.629	2.04721	4.41058	9.50231
8.59	73.7881	2.93087	9.26823	633.840	2.04801	4.41229	9.50600
8.60	73.9600	2.93258	9.27362	636.056	2.04880	4.41400	9.50969
8.61	74.1321	2.93428	9.27901	638.277	2.04959	4.41571	9.51337
8.62	74.3044	2.93598	9.28440	640.504	2.05039	4.41742	9.51705
8.63	74.4769	2.93769	9.28978	642.736	2.05118	4.41913	9.52073
8.64	74.6496	2.93939	9.29516	644.973	2.05197	4.42084	9.52441
8.65	74.8225	2.94109	9.30054	647.215	2.05276	4.42254	9.52808
8.66	74.9956	2.94279	9.30591	649.462	2.05355	4.42425	9.53175
8.67	75.1689	2.94449	9.31128	651.714	2.05434	4.42595	9.53542
8.68	75.3424	2.94618	9.31665	653.972	2.05513	4.42765	9.53908
8.69	75.5161	2.94788	9.32202	656.235	2.05592	4.42935	9.54274
8.70	75.6900	2.94958	9.32738	658.503	2.05671	4.43105	9.54640
8.71	75.8641	2.95127	9.33274	660.776	2.05750	4.43274	9.55006
8.72	76.0384	2.95296	9.33809	663.055	2.05828	4.43444	9.55371
8.73	76.2129	2.95466	9.34345	665.339	2.05907	4.43613	9.55736
8.74	76.3876	2.95635	9.34880	667.628	2.05986	4.43783	9.56101
8.75	76.5625	2.95804	9.35414	669.922	2.06064	4.43952	9.56466
8.76	76.7376	2.95973	9.35949	672.221	2.06143	4.44121	9.56830
8.77	76.9129	2.96142	9.36483	674.526	2.06221	4.44290	9.57194
8.78	77.0884	2.96311	9.37017	676.836	2.06299	4.44459	9.57557
8.79	77.2641	2.96479	9.37550	679.151	2.06378	4.44627	9.57921
8.80	77.4400	2.96648	9.38083	681.472	2.06456	4.44796	9.58284
8.81	77.6161	2.96816	9.38616	683.798	2.06534	4.44964	9.58647
8.82	77.7924	2.96985	9.39149	686.129	2.06612	4.45133	9.59009
8.83	77.9689	2.97153	9.39681	688.465	2.06690	4.45301	9.59372
8.84	78.1456	2.97321	9.40213	690.807	2.06768	4.45469	9.59734
8.85	78.3225	2.97489	9.40744	693.154	2.06846	4.45637	9.60095
8.86	78.4996	2.97658	9.41276	695.506	2.06924	4.45805	9.60457
8.87	78.6769	2.97825	9.41807	697.864	2.07002	4.45972	9.60818
8.88	78.8544	2.97993	9.42338	700.227	2.07080	4.46140	9.61179
8.89	79.0321	2.98161	9.42868	702.595	2.07157	4.46307	9.61540
8.90	79.2100	2.98329	9.43398	704.969	2.07235	4.46475	9.61900
8.91	79.3881	2.98496	9.43928	707.348	2.07313	4.46642	9.62260
8.92	79.5664	2.98664	9.44458	709.732	2.07390	4.46809	9.62620
8.93	79.7449	2.98831	9.44987	712.122	2.07468	4.46976	9.62980
8.94	79.9236	2.98998	9.45516	714.517	2.07545	4.47142	9.63339
8.95	80.1025	2.99166	9.46044	716.917	2.07622	4.47309	9.63698
8.96	80.2816	2.99333	9.46573	719.323	2.07700	4.47476	9.64057
8.97	80.4609	2.99500	9.47101	721.734	2.07777	4.47642	9.64415
8.98	80.6404	2.99666	9.47629	724.151	2.07854	4.47808	9.64774
8.99	80.8201	2.99833	9.48156	726.573	2.07931	4.47974	9.65132

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
9.00	81.0000	3.00000	9.48683	729.000	2.08008	4.48140	9.65489
9.01	81.1801	3.00167	9.49210	731.433	2.08085	4.48306	9.65847
9.02	81.3604	3.00333	9.49737	733.871	2.08162	4.48472	9.66204
9.03	81.5409	3.00500	9.50263	736.314	2.08239	4.48638	9.66561
9.04	81.7216	3.00666	9.50789	738.763	2.08316	4.48803	9.66918
9.05	81.9025	3.00832	9.51315	741.218	2.08393	4.48969	9.67274
9.06	82.0836	3.00998	9.51840	743.677	2.08470	4.49134	9.67630
9.07	82.2649	3.01164	9.52365	746.143	2.08546	4.49299	9.67986
9.08	82.4464	3.01330	9.52890	748.613	2.08623	4.49464	9.68342
9.09	82.6281	3.01496	9.53415	751.089	2.08699	4.49629	9.68697
9.10	82.8100	3.01662	9.53939	753.571	2.08776	4.49794	9.69052
9.11	82.9921	3.01828	9.54463	756.058	2.08852	4.49959	9.69407
9.12	83.1744	3.01993	9.54987	758.551	2.08929	4.50123	9.69762
9.13	83.3569	3.02159	9.55510	761.048	2.09005	4.50288	9.70116
9.14	83.5396	3.02324	9.56033	763.552	2.09081	4.50452	9.70470
9.15	83.7225	3.02490	9.56556	766.061	2.09158	4.50616	9.70824
9.16	83.9056	3.02655	9.57079	768.575	2.09234	4.50781	9.71177
9.17	84.0889	3.02820	9.57601	771.095	2.09310	4.50945	9.71531
9.18	84.2724	3.02985	9.58123	773.621	2.09386	4.51108	9.71884
9.19	84.4561	3.03150	9.58645	776.152	2.09462	4.51272	9.72236
9.20	84.6400	3.03315	9.59166	778.688	2.09538	4.51436	9.72589
9.21	84.8241	3.03480	9.59687	781.230	2.09614	4.51599	9.72941
9.22	85.0084	3.03645	9.60208	783.777	2.09690	4.51763	9.73293
9.23	85.1929	3.03809	9.60729	786.330	2.09765	4.51926	9.73645
9.24	85.3776	3.03974	9.61249	788.889	2.09841	4.52089	9.73996
9.25	85.5625	3.04138	9.61769	791.453	2.09917	4.52252	9.74348
9.26	85.7476	3.04302	9.62289	794.023	2.09992	4.52415	9.74699
9.27	85.9329	3.04467	9.62808	796.598	2.10068	4.52578	9.75049
9.28	86.1184	3.04631	9.63328	799.179	2.10144	4.52740	9.75400
9.29	86.3041	3.04795	9.63846	801.765	2.10219	4.52903	9.75750
9.30	86.4900	3.04959	9.64365	804.357	2.10294	4.53065	9.76100
9.31	86.6761	3.05123	9.64883	806.954	2.10370	4.53228	9.76450
9.32	86.8624	3.05287	9.65401	809.558	2.10445	4.53390	9.76799
9.33	87.0489	3.05450	9.65919	812.166	2.10520	4.53552	9.77148
9.34	87.2356	3.05614	9.66437	814.781	2.10595	4.53714	9.77497
9.35	87.4225	3.05778	9.66954	817.400	2.10671	4.53876	9.77846
9.36	87.6096	3.05941	9.67471	820.026	2.10746	4.54038	9.78195
9.37	87.7969	3.06105	9.67988	822.657	2.10821	4.54199	9.78543
9.38	87.9844	3.06268	9.68504	825.294	2.10896	4.54361	9.78891
9.39	88.1721	3.06431	9.69020	827.936	2.10971	4.54522	9.79239
9.40	88.3600	3.06594	9.69536	830.584	2.11045	4.54684	9.79586
9.41	88.5481	3.06757	9.70052	833.238	2.11120	4.54845	9.79933
9.42	88.7364	3.06920	9.70567	835.897	2.11195	4.55006	9.80280
9.43	88.9249	3.07083	9.71082	838.562	2.11270	4.55167	9.80627
9.44	89.1136	3.07246	9.71597	841.232	2.11344	4.55328	9.80974
9.45	89.3025	3.07409	9.72111	843.909	2.11419	4.55488	9.81320
9.46	89.4916	3.07571	9.72625	846.591	2.11494	4.55649	9.81666
9.47	89.6809	3.07734	9.73139	849.278	2.11568	4.55809	9.82012
9.48	89.8704	3.07896	9.73653	851.971	2.11642	4.55970	9.82357
9.49	90.0601	3.08058	9.74166	854.670	2.11717	4.56130	9.82703

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n^3	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$
9.50	90.2500	3.08221	9.74679	857.375	2.11791	4.56290	9.83048
9.51	90.4401	3.08383	9.75192	860.085	2.11865	4.56450	9.83392
9.52	90.6304	3.08545	9.75705	862.801	2.11940	4.56610	9.83737
9.53	90.8209	3.08707	9.76217	865.523	2.12014	4.56770	9.84081
9.54	91.0116	3.08869	9.76729	868.251	2.12088	4.56930	9.84425
9.55	91.2025	3.09031	9.77241	870.984	2.12162	4.57089	9.84769
9.56	91.3936	3.09192	9.77753	873.723	2.12236	4.57249	9.85113
9.57	91.5849	3.09354	9.78264	876.467	2.12310	4.57408	9.85456
9.58	91.7764	3.09516	9.78775	879.218	2.12384	4.57567	9.85799
9.59	91.9681	3.09677	9.79285	881.974	2.12458	4.57727	9.86142
9.60	92.1600	3.09839	9.79796	884.736	2.12532	4.57886	9.86485
9.61	92.3521	3.10000	9.80306	887.504	2.12605	4.58045	9.86827
9.62	92.5444	3.10161	9.80816	890.277	2.12679	4.58204	9.87169
9.63	92.7369	3.10322	9.81326	893.056	2.12753	4.58362	9.87511
9.64	92.9296	3.10483	9.81835	895.841	2.12826	4.58521	9.87853
9.65	93.1225	3.10644	9.82344	898.632	2.12900	4.58679	9.88195
9.66	93.3156	3.10805	9.82853	901.429	2.12974	4.58838	9.88536
9.67	93.5089	3.10966	9.83362	904.231	2.13047	4.58996	9.88877
9.68	93.7024	3.11127	9.83870	907.039	2.13120	4.59154	9.89217
9.69	93.8961	3.11288	9.84378	909.853	2.13194	4.59312	9.89558
9.70	94.0900	3.11448	9.84886	912.673	2.13267	4.59470	9.89898
9.71	94.2841	3.11609	9.85393	915.499	2.13340	4.59628	9.90238
9.72	94.4784	3.11769	9.85901	918.330	2.13414	4.59786	9.90578
9.73	94.6729	3.11929	9.86408	921.167	2.13487	4.59943	9.90918
9.74	94.8676	3.12090	9.86914	924.010	2.13560	4.60101	9.91257
9.75	95.0625	3.12250	9.87421	926.859	2.13633	4.60258	9.91596
9.76	95.2576	3.12410	9.87927	929.714	2.13706	4.60416	9.91935
9.77	95.4529	3.12570	9.88433	932.575	2.13779	4.60573	9.92274
9.78	95.6484	3.12730	9.88939	935.441	2.13852	4.60730	9.92612
9.79	95.8441	3.12890	9.89444	938.314	2.13925	4.60887	9.92950
9.80	96.0400	3.13050	9.89949	941.192	2.13997	4.61044	9.93288
9.81	96.2361	3.13209	9.90454	944.076	2.14070	4.61200	9.93626
9.82	96.4324	3.13369	9.90959	946.966	2.14143	4.61357	9.93964
9.83	96.6289	3.13528	9.91464	949.862	2.14216	4.61514	9.94301
9.84	96.8256	3.13688	9.91968	952.764	2.14288	4.61670	9.94638
9.85	97.0225	3.13847	9.92472	955.672	2.14361	4.61826	9.94975
9.86	97.2196	3.14006	9.92975	958.585	2.14433	4.61983	9.95311
9.87	97.4169	3.14166	9.93479	961.505	2.14506	4.62139	9.95648
9.88	97.6144	3.14325	9.93982	964.430	2.14578	4.62295	9.95984
9.89	97.8121	3.14484	9.94485	967.362	2.14651	4.62451	9.96320
9.90	98.0100	3.14643	9.94987	970.299	2.14723	4.62607	9.96655
9.91	98.2081	3.14802	9.95490	973.242	2.14795	4.62762	9.96991
9.92	98.4064	3.14960	9.95992	976.191	2.14867	4.62918	9.97326
9.93	98.6049	3.15119	9.96494	979.147	2.14940	4.63073	9.97661
9.94	98.8036	3.15278	9.96995	982.108	2.15012	4.63229	9.97996
9.95	99.0025	3.15436	9.97497	985.075	2.15084	4.63384	9.98331
9.96	99.2016	3.15595	9.97998	988.048	2.15156	4.63539	9.98665
9.97	99.4009	3.15753	9.98499	991.027	2.15228	4.63694	9.98999
9.98	99.6004	3.15911	9.98999	994.012	2.15300	4.63849	9.99333
9.99	99.8001	3.16070	9.99500	997.003	2.15372	4.64004	9.99667

TABLE III — IMPORTANT NUMBERS

A. *Units of Length*

English Units		Metric Units	
12 inches (in.)	= **1 foot** (ft.)	10 millimeters (mm.)	= **1 centimeter** (cm.)
3 feet	= 1 yard (yd.)	10 centimeters	= 1 decimeter (dm.)
$5\frac{1}{2}$ yards	= 1 rod (rd.)	10 decimeters	= **1 meter** (m.)
320 rods	= **1 mile** (mi.)	10 meters	= 1 dekameter (Dm.)
		1000 meters	= **1 kilometer** (Km.)

English to Metric	Metric to English
1 in. = 2.5400 cm.	**1 cm. = 0.3937 in.**
1 ft. = 30.480 cm.	1 m. = 39.37 in. = 3.2808 ft.
1 mi. = 1.6093 Km.	1 Km. = 0.6214 mi.

B. *Units of Area or Surface*

1 square yard = 9 square feet = 1296 square inches
1 acre (A.) = 160 square rods = 4840 square yards
1 square mile = 640 acres = 102400 square rods

C. *Units of Measurement of Capacity*

Dry Measure		Liquid Measure	
2 pints (pt.)	= 1 quart (qt.)	4 gills (gi.)	= 1 pint (pt.)
8 quarts	= 1 peck (pk.)	2 pints	= 1 quart (qt.)
4 pecks	= 1 bushel (bu.)	4 quarts	= 1 gallon (gal.)
		1 gallon	= 231 cu. in.

D. *Metric Units to English Units*

1 liter = 1000 cu. cm. = 61.02 cu. in. = 1.0567 liquid quarts
1 quart = .94636 liter = 946.36 cu. cm.
1000 grams = 1 kilogram (Kg.) = 2.2046 pounds (lb.)
1 pound = .453593 kilogram = 453.59 grams

E. *Other Numbers*

π = ratio of circumference to diameter of a circle
= 3.14159265
1 radian = angle subtended by an arc equal to the radius
= $57° 17' 44''.8 = 57°.2957795 = 180°/\pi$
1 degree = 0.01745329 radian, or $\pi/180$ radians
Weight of 1 cu. ft. of water = 62.425 lb.

SYMBOLS AND ABBREVIATIONS

The following symbols and abbreviations are used for the sake of brevity throughout the present book :

$=$	equal, or is equal to
$\neq$	not equal, or is not equal to
$>$	greater than
$<$	less than
$\cong$	is congruent to
$\perp$	perpendicular, or is perpendicular to
$\parallel$	parallel, or is parallel to
$\sim$	similar, or is similar to
$\angle$	angle
$\angle$s	angles
$\triangle$	triangle
$\triangle$s	triangles
▱	parallelogram
▱s	parallelograms
○	circle
Ⓢ	circles
$\frown$	arc

Ax.	Axiom
Cons.	Construction, or by construction
Cor.	Corollary
Def.	Definition
Hyp.	Hypothesis, or by hypothesis
Iden.	being identical
Prop.	Proposition
rt.	right
st.	straight
Th.	Theorem
Prob.	Problem
Fig.	Figure or diagram
. . . .	*and so on*
$\therefore$	*hence* or *therefore*

The signs $+$, $-$, $\times$, $\div$, are used with the same meanings as in algebra. The following agreements are also made :

$$a \times b = a \cdot b = ab, \quad a \div b = a/b = a : b$$

SYLLABUS OF PLANE GEOMETRY

INTRODUCTION

PART I. DRAWING SIMPLE FIGURES

PART II. FUNDAMENTAL IDEAS

PART III. STATEMENTS FOR REFERENCE

27. Axioms.

1. *If equals are added to equals, the sums are equal.* Thus, if $a = b$ and $c = d$, then $a + c = b + d$.

2. *If equals are subtracted from equals, the remainders are equal.* Thus, if $a = b$ and $c = d$, then $a - c = b - d$.

3. *If equals are multiplied by equals, the products are equal.* Thus, if $a = b$ and $c = d$, then $ac = bd$.

4. *If equals are divided by equals, the quotients are equal.* Thus, if $a = b$ and $c = d$, then $\frac{a}{c} = \frac{b}{d}$. In applying this axiom it is supposed that c and d are *not equal to zero.*

5. *If equals are added to unequals, the results are unequal and in the same order.* Thus, if $a = b$ and $c > d$, then $a + c > b + d$.

6. *If equals are subtracted from unequals, the results are unequal and in the same order.* Thus, if $a > b$ and $c = d$, then $a - c > b - d$.

7. *If unequals are added to unequals in the same sense, the results are unequal in the same order.* Thus, if $a > b$ and $c > d$, then $a + c > b + d$.

8. *If unequals are subtracted from equals, the results are unequal in the opposite order.* Thus, if $a = b$ and $c > d$, then $a - c < b - d$.

9. *Quantities equal to the same quantity, or to equal quantities, are equal to each other.* In other words, *a quantity may be* ***substituted for*** *its equal at any time in any expression.*

* This syllabus contains the principal statements of fact, with their article numbers, from the *Plane Geometry* by the authors of this book.

10. *The whole of a quantity is greater than any one of its parts.*

11. *The whole of a quantity is equal to the sum of its parts.*

28. Postulates.

1. *Only one straight line can be drawn joining two given points.*

2. *A straight line can be extended indefinitely.*

3. *A straight line is the shortest curve that can be drawn between two points.*

4. *A circle can be described about any point as a center and with a radius of any length.*

5. *A figure can be moved unaltered to a new position.*

6. *All straight angles are equal.* Hence, also, *all right angles are equal*, for a right angle is half of a straight angle.

30. Preliminary Construction Problems.

1. To construct a triangle, each of whose sides is equal to a given length. § 3.

2. To construct a triangle, whose three sides are, respectively, equal to three given lengths. § 4.

3. To construct a perpendicular to a given straight line at a given point in that line. § 5.

4. To construct a perpendicular to a given line from a given point not on that line. § 6.

5. To construct, at a given point in a given line, another line that makes an angle equal to a given angle with the given line. § 7.

6. To divide a portion of a straight line into two equal parts. (To bisect a line.) § 8.

7. To divide a given angle into two equal parts. (To bisect an angle.) § 9.

31. Preliminary Theorems.

1. All radii of the same circle are equal. § 2; and § 23.

2. Circles whose radii are equal can be placed upon each other so that their centers and their circumferences coincide (lie exactly upon each other).

3. Equal angles may be placed upon each other so that their vertices coincide and their corresponding sides fall along the same straight lines. This is, in fact, what we mean by *equal* angles.

4. Two straight lines have at most one point in common. See postulate 1, § 28.

5. Two circles have at most two points in common. See § 8.

6. A straight line and a circle may have at most two points in common.

7. At a given point in a given line only one perpendicular can be drawn to that line. (A consequence of Problem 3, § 5.)

8. Complements of the same angle, or of equal angles, are equal.

9. Supplements of the same angle, or of equal angles, are equal.

10. Vertical angles are equal. § 19.

11. If two adjacent angles have their exterior sides in a straight line, they are supplementary. § 18.

12. If two adjacent angles are supplementary, they have their exterior sides in a straight line.

13. If each of two figures can be placed upon a third figure so as to coincide with it, they can be placed upon each other so that they coincide.

14. Any desired angle may be drawn, and any angle may be measured, by the use of a protractor. (But the use of this instrument is not permitted when a figure is to be *constructed*. See § 20.)

15. A perpendicular to a given line through any given point may be drawn by means of a set square or a drawing triangle. (But the use of these instruments is not permitted when a figure is to be *constructed*. See § 20.)

16. The area of a rectangle (in terms of a unit square) is equal to the product of its width and its height, measured in units of length equal to one side of the unit square.

17. The area of any given figure is greater than the area of any figure that is drawn completely within it.

18. The areas of two figures are equal if they consist of corresponding portions that can be made to coincide.

CHAPTER I

RECTILINEAR FIGURES

PART I. TRIANGLES

35. Theorem I. *If two triangles have two sides and the included angle of the one equal, respectively, to two sides and the included angle of the other, the triangles are congruent.*

36. Corollary 1. *Two right triangles are congruent if the two sides of the one are equal, respectively, to the two sides of the other.*

37. Theorem II. *If two triangles have two angles and the included side in the one equal, respectively, to two angles and the included side in the other, the triangles are congruent.*

38. Corollary 1. *Two right triangles are congruent if an acute angle and its adjacent side in one are equal, respectively, to an acute angle and its adjacent side in the other.*

40. Theorem III. *In an isosceles triangle the angles opposite the equal sides are equal.*

41. Corollary 1. *If a triangle is equilateral, it is also equiangular.*

43. Theorem IV. *The bisector of the angle at the vertex of an isosceles triangle is perpendicular to the base and bisects the base.*

44. Corollary 1. *In any isosceles triangle* (*a*) *The bisector of the angle at the vertex divides the triangle into two congruent right triangles.*

(*b*) *The bisector of the vertical angle coincides with both the altitude and the median drawn through the vertex.*

(*c*) *The perpendicular bisector of the base passes through the vertex, and divides the triangle into two congruent right triangles.*

45. Theorem V. *If two triangles have three sides of the one equal, respectively, to the three sides of the other, they are congruent.*

46. Corollary 1. *Three sides determine a triangle; that is, if the three sides are given, the triangle is thereby fixed.*

47. Theorem VI. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

PART II. PARALLEL LINES

49. Parallel Postulate. ***Only one line can be drawn through a given point parallel to a given line.***

50. Corollary 1. *Lines parallel to the same line are parallel to each other.*

51. Theorem VII. ***When two lines are cut by a transversal, if the alternate interior angles are equal, the two lines are parallel.***

52. Corollary 1. *Lines perpendicular to the same line are parallel.*

54. Theorem VIII. ***If two parallel lines are cut by a transversal, the alternate interior angles are equal.***

56. Theorem IX. *If two lines are cut by a transversal and the corresponding angles are equal, the lines are parallel.*

57. Corollary 1. *If two lines are cut by a transversal and the two interior angles on the same side of the transversal are supplementary, the lines are parallel.*

58. Corollary 2. *From a given point only one perpendicular can be drawn to a given line.*

59. Theorem X. (*Converse of Theorem IX.*) *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

60. Corollary 1. *If a line is perpendicular to one of two parallels, it is perpendicular to the other also.*

61. Problem 1. *To construct a line parallel to a given line and passing through a given point.*

PART III. ANGLES AND TRIANGLES

62. Theorem XI. ***The sum of three angles of any triangle is equal to two right angles, or 180°.***

63. Corollary 1. ***The sum of the two acute angles of any right triangle is one right angle, or 90°.***

64. Corollary 2. ***An exterior angle of any triangle is equal to the sum of its opposite interior angles.***

65. Corollary 3. *Each angle of an equilateral triangle is equal to 60°.*

66. Corollary 4. *If two angles of one triangle are equal, respectively, to two angles of another triangle, then the third angles are likewise equal.*

67. Theorem XII. *Two angles whose sides are respectively parallel are either equal or supplementary.*

68. Theorem XIII. *Two angles whose sides are respectively perpendicular to each other are either equal or supplementary.*

69. Theorem XIV. ***Two right triangles are congruent if the hypotenuse and an acute angle of the one are equal, respectively, to the hypotenuse and an acute angle of the other.***

70. Theorem XV. ***Two right triangles are congruent if the hypotenuse and a side of the one are equal, respectively, to the hypotenuse and a side of the other.***

71. Corollary 1. *If two oblique lines of equal length are drawn from a point C in a perpendicular CD to a line AB, they cut off equal distances from the foot of the perpendicular, and conversely.*

72. Theorem XVI. (*Converse of Theorem III.*) *If two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.*

73. Corollary 1. *An equiangular triangle is also equilateral.*

74. Corollary 2. *If two oblique lines are drawn from a point C in a perpendicular CD to a line AB, so as to make equal angles with AB, they are equal, and conversely.*

75. Theorem XVII. *If two angles of a triangle are unequal, the sides opposite them are unequal and the greater side is opposite the greater angle.*

76. Corollary 1. *If two triangles have two sides of the one equal to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*

77. Corollary 2. *If two oblique lines are drawn from a point C in a perpendicular CD to a line AB, and if the base angles at A and B are unequal, the oblique line opposite the greater base angle is the greater;* in particular, *the perpendicular CD is itself the shortest line from C to any point of AB.*

78. Theorem XVIII. *If two sides of a triangle are unequal, the angles opposite them are unequal and the greater angle is opposite the greater side.*

79. Corollary 1. *If two triangles have two sides of the one equal to two sides of the other, but the third side of the first greater than the third side of the second, then the included angle of the first is greater than the included angle of the second.*

80. Corollary 2. *If from a point C in a perpendicular CD to a line AB unequal oblique lines are drawn to the base AB, the longer of the oblique lines is opposite the larger of the two base angles.*

PART IV. QUADRILATERALS

82. Theorem XIX. ***Either diagonal of a parallelogram divides it into two congruent triangles.***

83. Corollary 1. *Any side of a parallelogram is equal to the side opposite it.*

84. Corollary 2. *The segments of parallel lines included between parallel lines are equal.*

85. Theorem XX. *If a quadrilateral has both pairs of opposite sides equal, it is a parallelogram.*

86. Theorem XXI. *If a quadrilateral has one pair of sides equal and parallel, it is a parallelogram.*

87. Theorem XXII. *The diagonals of a parallelogram bisect each other.*

88. Theorem XXIII. *Two parallelograms are congruent if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.*

89. Theorem XXIV. ***The line joining the middle points of the two sides of a triangle is parallel to the base and equal to half the base.***

90. Corollary 1. (*Converse of § 89.*) *The line drawn through the middle point of one side of a triangle parallel to the base bisects the other side.*

91. Theorem XXV. ***If three parallel lines cut off two equal portions of one transversal, they cut off two equal portions of any other transversal.***

92. Corollary 1. *If a series of parallel lines cut off equal portions of one transversal, they cut off equal portions of any other transversal.*

93. Corollary 2. *If three parallel lines cut off two portions of one transversal, one of which is double the other, they cut off two portions of any other transversal, one of which is double the other.*

94. Corollary 3. *If three parallel lines cut off two portions of one transversal, one of which is n times the other, they cut off two portions of any other transversal, one of which is n times the other.*

PART V. POLYGONS

97. Theorem XXVI. *The sum of the interior angles of a polygon is two right angles taken as many times as the figure has sides, less two.*

98. Theorem XXVII. *The sum of the exterior angles of a polygon formed by producing the sides in succession is equal to four right angles.*

PART VI. THE LOCUS OF A POINT

100. Theorem XXVIII. ***The locus of all points equidistant from the extremities of a line is the perpendicular bisector of that line.***

101. Theorem XXIX. ***The bisector of an angle is the locus of all points equidistant from its sides.***

102. Supplementary Propositions on Altitudes, Medians, etc.

Theorem XXX. *The perpendiculars erected at the middle points of the sides of a triangle meet in a point.*

Theorem XXXI. *The bisectors of the angles of a triangle meet in a point.*

Theorem XXXII. *The altitudes of a triangle meet in a point.*

Theorem XXXIII. *The medians of a triangle meet in a point which is two thirds of the distance from any vertex to the middle point of the opposite side.*

CHAPTER II

THE CIRCLE

PART I. CHORDS. ARCS. CENTRAL ANGLES

104. Postulates.

(1) *As a central angle increases, its intercepted arc increases, and vice versa;* and *as a central angle decreases, its intercepted arc decreases, and vice versa.*

(2) *In the same circle (or equal circles), equal central angles intercept equal arcs;* and *equal arcs subtend equal central angles.*

106. Theorem I. *In the same circle (or in equal circles), equal arcs subtend equal chords.*

107. Theorem II. (*Converse of* § 106.) *In the same circles (or in equal circles), equal chords subtend equal arcs.*

108. Theorem III. ***A diameter perpendicular to a chord bisects the chord and the arc subtended by it.***

109. Theorem IV. *In the same circle (or in equal circles), equal chords are equally distant from the center, and, conversely, chords that are equally distant from the center are equal.*

110. Theorem V. *In the same circle (or in equal circles), if two unequal chords are drawn, the longer one is nearer the center.*

111. Corollary 1. (*Converse of* § 110.) *In the same circle (or in equal circles), if two chords are unequally distant from the center, the more remote is the less.*

PART II. TANGENTS AND SECANTS

115. Theorem VI. ***A line perpendicular to a radius at its extremity is tangent to the circle.***

116. Corollary 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

117. Corollary 2. *A perpendicular to a tangent at its point of contact passes through the center of the circle.*

118. Theorem VII. *Two tangents drawn to a circle from a point outside are of equal length.*

119. Theorem VIII. *Two parallel lines intercept equal arcs on a circle.*

120. Theorem IX. *Through three given points not all on the same straight line, one and only one circle can be drawn.*

121. Corollary 1. *A circle may be drawn to circumscribe any triangle.*

122. Corollary 2. *The perpendicular bisectors of the sides of a triangle meet in a point.*

123. Corollary 3. *A circle may be completed if any arc of it is given.*

124. Theorem X. *A circle may be* ***inscribed*** *in any triangle.*

125. Corollary 1. *A circle drawn from any point on the bisector of an angle, with a radius equal to the distance from that point to one side, is tangent to both sides of the angle.*

PART III. MEASUREMENT OF ANGLES

130. Theorem XI. *In the same circle (or in equal circles), two central angles have the same ratio as their intercepted arcs.*

NOTE 1. We may assume as a **postulate** that *if two geometric ratios are equal whenever their terms are commensurable, they are equal also when their terms are incommensurable.*

NOTE 2. ***A central angle is measured by its intercepted arc.***

132. Theorem XII. ***An inscribed angle is measured by one half of its intercepted arc.***

133. Corollary 1. *Any angle inscribed in a semicircle is a right angle.*

134. Corollary 2. *Any angle inscribed in a segment greater than a semicircle is acute, while any angle inscribed in a segment less than a semicircle is obtuse.*

135. Corollary 3. *All angles inscribed in the same segment are equal.*

136. Theorem XIII. *An angle formed by two chords intersecting within a circle is measured by one half of the sum of the intercepted arcs.*

137. Theorem XIV. *An angle formed by a tangent and a chord drawn through the point of tangency is measured by one half of the intercepted arc.*

138. Theorem XV. *An angle formed by two secants, or by a tangent and a secant, or by two tangents that meet outside a circle is measured by one half the difference of the intercepted arcs.*

PART IV. CONSTRUCTION PROBLEMS

139. Problem 1. *Through a given point to draw a tangent to a circle.*

140. Problem 2. *To circumscribe a circle about a given triangle.*

141. Problem 3. *To inscribe a circle in a given triangle.*

142. Problem 4. *On a given straight line to construct a segment of a circle that shall contain a given angle.*

CHAPTER III

PROPORTION. SIMILARITY

PART I. GENERAL THEOREMS ON PROPORTION

144. General Theorems on Proportion.

Theorem A. *In any proportion, the product of the extremes is equal to the product of the means.*

Corollary 1. *If the two antecedents of a proportion are equal, the consequents are also equal.*

Theorem B. *If the product of two numbers is equal to the product of two other numbers, either pair may be made the means of a proportion in which the other two are taken as the extremes.*

Theorem C. *If four quantities are in proportion, they are in proportion by **inversion**.*

Theorem D. *If four quantities are in proportion, they are in proportion by **alternation**.*

Theorem E. *If four quantities are in proportion, they are in proportion by **composition**.*

Theorem F. *If four quantities are in proportion, they are in proportion by **division**.*

Theorem G. *If four quantities are in proportion, they are in proportion by composition and division.*

Theorem H. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

PART II. PROPORTIONAL LINE-SEGMENTS

145. Theorem I. ***A line parallel to the base of a triangle divides the other sides proportionally.***

146. Corollary 1. *If a line is drawn parallel to the base of a triangle, either side is to one of its segments as the other side is to its corresponding segment.*

147. Theorem II. (*Converse of Theorem I.*) *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*

148. Corollary 1. *If a line cuts two sides of a triangle in such a way that either side is to one of its segments as the other side is to its corresponding segment, then the line is parallel to the third side.*

149. Theorem III. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the sides of the angle.*

150. Theorem IV. ***If a series of parallels be cut by two lines, the corresponding segments are proportional.***

151. Problem 1. *To divide a given line into parts proportional to any number of given lines.*

153. Problem 2. *To find the fourth proportional to three given lines.*

PART III. SIMILAR TRIANGLES AND POLYGONS

155. Theorem V. ***If two triangles are mutually equiangular, they are similar.***

156. Corollary 1. *Two triangles are similar if two angles of the one are equal respectively to two angles of the other.*

157. Corollary 2. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

158. Theorem VI. (*Converse of Theorem V.*) *If two triangles are similar, they are mutually equiangular.*

159. Theorem VII. *Two triangles are similar if an angle of the one equals an angle of the other and the including sides are proportional.*

161. Theorem VIII. ***If, in any right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the two right triangles thus formed are similar to each other and to the given triangle.***

162. Corollary 1. *In any right triangle the perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse.*

163. Corollary 2. *If, in any right triangle, a perpendicular is drawn from the vertex of the right angle to the hypotenuse, each side of the right triangle is the mean proportional between the hypotenuse and the segment adjacent to that side.*

164. Problem 3. *To find the mean proportional between two lines.*

166. Theorem IX. *Regular polygons of the same number of sides are similar.*

167. Theorem X. *The perimeters of two similar polygons are to each other in the same ratio as any two corresponding sides.*

PART IV. PROPORTIONAL PROPERTIES OF CHORDS, SECANTS, AND TANGENTS

168. Theorem XI. *If two chords intersect within a circle, the product of the segments of the one is equal to the product of the segments of the other.*

169. Theorem XII. *If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the entire secant and its exterior segment.*

170. Theorem XIII. *If from a fixed point without a circle any two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*

172. Problem 4. *To divide a given line segment in extreme and mean ratio.*

PART V. SIMILAR RIGHT TRIANGLES. TRIGONOMETRIC RATIOS

174. Theorem XIV. ***If two right triangles have one acute angle of one equal to one acute angle of the other, their corresponding sides are in the same ratios.***

175. Corollary 1. *If an acute angle of a right triangle is known, the ratios of the sides are all determined.*

176. Corollary 2. *If the ratio of any pair of sides of a right triangle is given, the acute angles are determined.*

179. Theorem XV. *Corresponding altitudes divide any two similar triangles into two corresponding pairs of similar right triangles.*

180. Corollary 1. *Any two similar polygons may be subdivided into corresponding pairs of similar right triangles.*

CHAPTER IV

AREAS OF POLYGONS. PYTHAGOREAN THEOREM

181. Area of a Rectangle. ***The area of a rectangle is equal to the product of its base by its height.***

182. Corollary 1. *The area of a square is equal to the square of its side.*

183. Corollary 2. *The areas of two rectangles are to each other as the products of their bases and altitudes.*

184. Corollary 3. *Two rectangles that have equal altitudes are to each other as their bases; two rectangles that have equal bases are to each other as their altitudes.*

186. Theorem I. ***The area of a parallelogram is equal to the product of its base by its altitude.***

187. Corollary 1. (*a*) *Two parallelograms are to each other as the products of their bases and altitudes.*

(*b*) *Two parallelograms that have equal bases and equal altitudes are equal in area.*

188. Corollary 2. *Two parallelograms that have equal altitudes are to each other as their bases; two parallelograms that have equal bases are to each other as their altitudes.*

189. Theorem II. ***The area of a triangle is equal to one half the product of its base by its altitude.***

190. Corollary 1. (*a*) *Two triangles are to each other as the products of their bases and altitudes.*

(*b*) *Two triangles that have equal bases are to each other as their altitudes.*

(*c*) *Two triangles that have equal altitudes are to each other as their bases.*

(*d*) *Two triangles that have equal bases and equal altitudes are equal in area.*

191. Theorem III. *The area of a trapezoid is equal to the product of its altitude and one half the sum of its bases.*

192. Corollary 1. *The area of a trapezoid is equal to the product of its altitude and the line joining the mid-points of the non-parallel sides.*

The area of a trapezoid is equal to the product of its altitude and its median.

193. Theorem IV. *Two triangles that have an acute angle of the one equal to an acute angle of the other are to each other as the products of the sides including the equal angles.*

194. Theorem V. *Similar triangles are to each other as the squares of any two corresponding sides.*

195. Corollary 1. *The areas of two similar polygons are to each other as the squares of any two corresponding sides.*

The areas of two similar polygons are to each other as the squares of any two corresponding lines.

196. Theorem VI. The Pythagorean Theorem. ***The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the two sides.***

197. Corollary 1. *The square on either side of a right triangle is equivalent to the square on the hypotenuse diminished by the square on the other side.*

199. Theorem VII. *In any triangle the square on the side opposite an* **acute** *angle is equal to the sum of the squares on the other two sides diminished by twice the product of one of those sides and the projection of the other upon it.*

200. Theorem VIII. *In any obtuse triangle the square on the side opposite the* **obtuse** *angle is equal to the sum of the squares on the other sides increased by twice the product of one of those sides and the projection of the other upon it.*

201. Problem 1. *To construct a square whose area shall be equal to the sum of the areas of two given squares.*

202. Problem 2. *To construct a triangle whose area shall be equal to that of a given polygon.*

CHAPTER V

REGULAR POLYGONS AND CIRCLES

204. Theorem I. *If a circle is divided into a number of equal arcs:*

(*a*) *the chords joining the points of division form a regular inscribed polygon;*

(*b*) *tangents drawn at the points of division form a regular circumscribed polygon.*

205. Theorem II. (*a*) *A circle may be circumscribed about any regular polygon;* (*b*) *a circle may also be inscribed in it.*

207. Theorem III. *The area of a regular polygon is equal to half the product of its apothem and perimeter.*

210. Areas and Lengths of Circles.

If the number of sides of the regular inscribed and regular circumscribed polygons is repeatedly doubled:

(*a*) *their areas approach the area of the circle as a common limit;*

(*b*) *their perimeters approach the length of the circumference of the circle as a common limit.*

211. Theorem IV. ***The circumferences of two circles are to each other as their radii.***

212. Corollary 1. *The ratio of a circumference to its diameter is the same for all circles.*

213. The Number π. *The number obtained by dividing the circumference of any circle by its diameter is denoted by the Greek letter* π.

214. Corollary 2. ***In any circle*** $c = \pi d$, ***or*** $c = 2\pi r$, *where* r *is the radius,* d *the diameter, and* c *the length of the circumference.*

215. Theorem V. ***The area of a circle is equal to one half the product of its radius and its circumference.***

216. Corollary 1. ***The area of a circle is equal to π times the square of its radius,*** *that is,* $A = \pi r^2$.

217. Corollary 2. *The areas of two circles are to each other as the squares of their radii.*

220. Problem 1. *Given the side and radius of a regular inscribed polygon, to find the side of a regular inscribed polygon of double the number of sides.*

221. Corollary 1. *If* $r = 1$, *and* $s =$ *the side of the inscribed polygon, the side* AC *of the regular inscribed polygon of double the number of sides is,*

$$AC = \sqrt{2 - \sqrt{4 - s^2}}.$$

222. Problem 2. *To compute approximately the value of* π.

$\pi = \frac{6.28317}{2}$ (approximately) $= 3.14159$ (usually written 3.1416, or $3\frac{1}{7}$).

223. Problem 3. *To inscribe a square in a given circle.*

224. Problem 4. *To inscribe a regular hexagon in a given circle.*

225. Problem 5. *To inscribe a regular decagon in a given circle.*

APPENDIX TO PLANE GEOMETRY

MAXIMA AND MINIMA

226. *Of all chords through* P, *the diameter* PQ *is the* **maximum** *(greatest).*

Of all regular polygons inscribed in a circle, the equilateral triangle has the **minimum** *(least) area, or simply, is the* **minimum**.

Of all the straight lines that can be drawn from a fixed point to a given line, the perpendicular is the **minimum**.

227. Theorem I. *Of all triangles that have the same two given sides, that in which these sides include a right angle is the maximum.*

229. Theorem II. *Of all isoperimetric triangles having the same base, the isosceles is the maximum.*

230. Corollary 1. *Of all isoperimetric triangles, the equilateral is the maximum.*

231. Theorem III. *Of all isoperimetric polygons having the same number of sides, the maximum is equilateral.*

232. Theorem IV. *Of all polygons with sides all given but one, the maximum (in area) can be inscribed in a semicircle having the undetermined side for its diameter.*

233. Theorem V. *Of all polygons with the same given sides, that which can be inscribed in a circle is the maximum.*

234. Corollary 1. *Of all isoperimetric polygons of a given number of sides, the maximum is regular.*

235. Theorem VI. *Of two isoperimetric regular polygons, the one having the greater number of sides has the greater area.*

236. Corollary 1. *The circle is the maximum of all isoperimetric plane closed figures.*

237. Theorem VII. *Of all regular polygons of the same area, that which has the greatest number of sides has the minimum perimeter.*

238. Corollary 1. *Of all plane closed figures that are equal in area, the circle has the minimum perimeter.*

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